Local minimum spanning tree optimization

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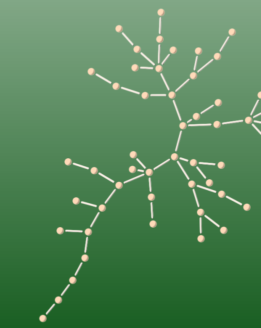
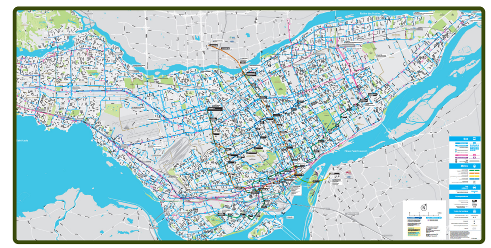


Motivation

Imagine a city manager trying to improve the global state of the transportation system of a city, from building new roads, to changing bus routes, or even adding new public transit. The city is currently laid out in a way that does not necessarily match the population and its organisation and they might want to optimize its structure while affecting the least people. Given this problem, how local can the changes be so that they lead to the desired global optimum?

Although we do not directly address this exact optimization problem, our problem falls under the same framework, wondering how local optimizations can lead to the global optimum.



Can we optimize Montreal's traffic while minimizing its constructions?

Optimizations

Our work focusses on studying properties of *optimizations*, as defined below.

Definition [Optimization]. *Given a complete weighted graph (K_n, w), an optimization $\mathcal{X} = (H_0, S_0, H_1, S_2, \dots, H_m)$ is an alternating sequence where:*

- *H_i is a spanning subgraph of K_n;*
- *S_i is a connected subset of H_i;*
- *H_{i+1} is obtained from H_i by replacing its induced subgraph $H_i[S_i]$ on S_i by its corresponding minimum spanning tree.*

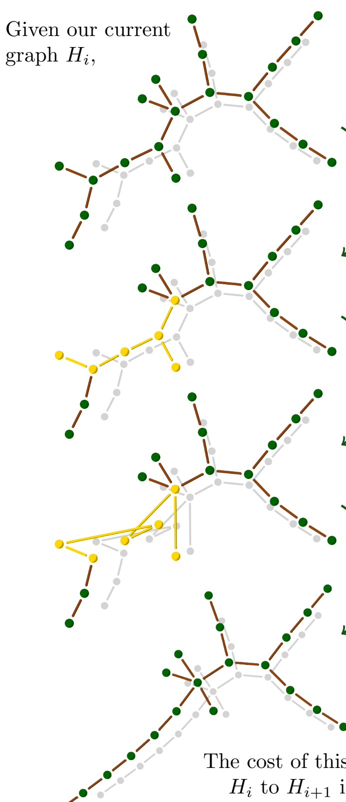
Since the weights of $H_0, H_1, \dots, H_m$ decreases, we are mainly interested in optimizations that eventually reach the minimum spanning tree. Write $\mathcal{O}^+(H_0)$ for the set of such optimizations, where H_m is the minimum spanning tree of K_n.

Given an optimization $\mathcal{X} = (H_0, S_0, \dots, H_m)$, we want to define a measure on its efficiency. To do so, we define the cost of an optimization to be the maximal weight of the replaced subgraph when going from H_i to H_{i+1}:

$$\mathrm{cost}(\mathcal{X}) := \max_{0 \le i < m} \Big\{ w\big(H_i[S_i]\big) \Big\} .$$

In the case of the city manager problem, an optimization corresponds to a sequence of local changes implemented to improve the road layout, and the cost of an optimization corresponds to the largest size of any single change.

Example



Results

Under mild assumptions on (K_n, w) and H_0, the minimal value of the cost to reach the minimum spanning tree starting from H_0 nicely converges when $n \to \infty$, as stated in the following theorem.

Theorem [Addario-Berry, Barrett, and C.]. *Let (K_n, w) be the complete graph with independent* Uniform($[0,1]$) *edge weights and let H_0 be a spanning subgraph of K_n chosen independently of w. Then, as $n \to \infty$,*

$$\min \Big\{ \mathrm{cost}(\mathcal{X}) : \mathcal{X} \in \mathcal{O}^+(H_0) \Big\} \xrightarrow{\mathbb{P}} 1 .$$

The fact that this minimum is larger than $1 - o(1)$ is not difficult to check, simply by considering the largest edge in H_0. The interesting part of this result is the upper bound, stating that there actually exists optimizations whose cost is $1 + o(1)$. Below is a quick explanation of our main argument.

Method

The main method of our proof consists of finding an optimization with small cost able to go from the minimum spanning tree on a graph of size k to the minimum spanning tree on a graph of size $k+1$.

Assume that we managed to obtain H_i such that there exists $|S| = k$ with $H_i[S]$ being the minimum spanning tree on S. Then, by choosing $S_i = S \cup \{v\}$, where v is a neighbour of S in H_i, it follows that H_{i+1} contains the minimum spanning tree on $S \cup \{v\}$. Moreover, the weight of such transition is

$$w\big(H_i[S_i]\big) \simeq 1 + w(\mathrm{MST}_k) \simeq 1 + \zeta(3) \simeq 2.2 \dots$$

where the 1 comes from the maximal possible weight from S to v and $\zeta(3)$ comes from the asymptotic weight of large minimum spanning trees. This method gives an upper bound of $\simeq 2.2$ for the minimal cost, which is not tight. However a similar but finer approach leads to the desired $1 + o(1)$.

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