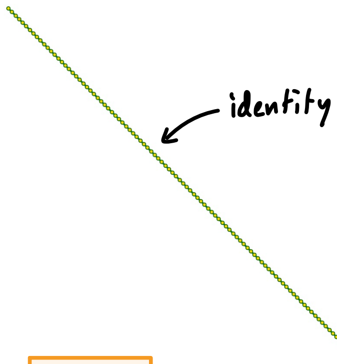
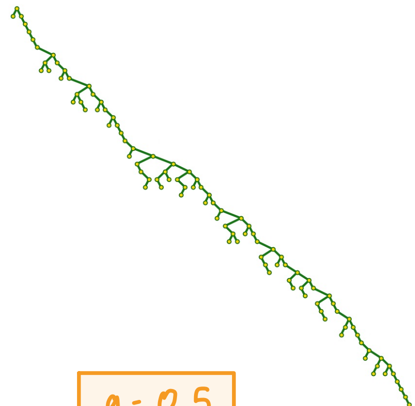

The height of random trees
The height of Mallows trees
(Part II)

Examples of Mallows trees (n,q):

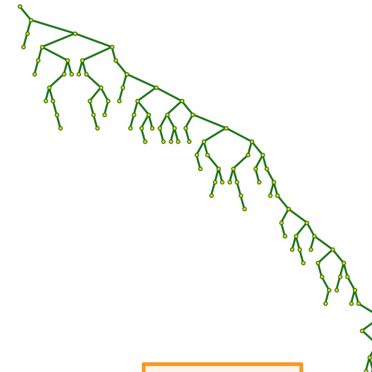
$n = 100$



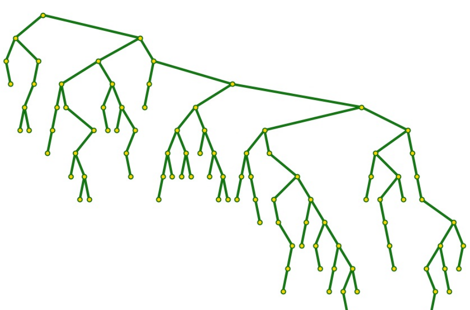
$q = 0$



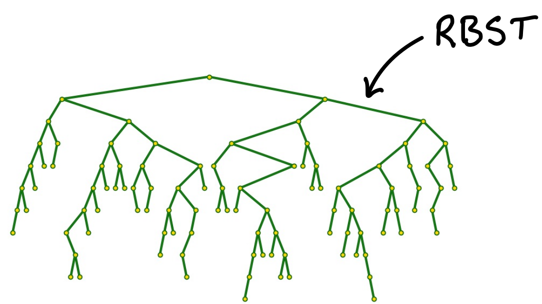
$q = 0.5$



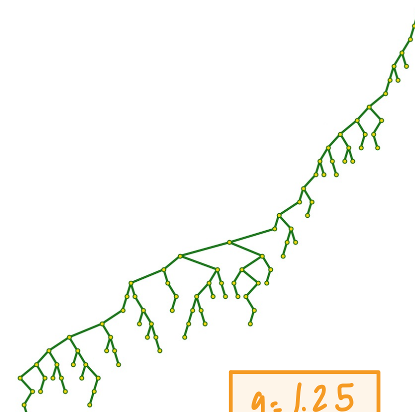
$q = 0.8$



$q = 0.9$



$q = 1$



$q = 1.25$

1. Mallows permutations
2. Results on the height
3. Proof techniques

1 Mallows permutations

- Mallows permutations are random permutations which generalize uniform permutations.
- They depend on a parameter $q \in [0, \infty)$ creating a bias
- When $q < 1$, tend to be in increasing order.
  When $q > 1$, tend to be in decreasing order.

1 Mallows permutations

Definition

For $n \in \mathbb{N}$, $q \in [0, \infty)$, the Mallows distribution $\pi_{n,q}$ is defined by:

$$\pi_{n,q}(\sigma) \propto q^{\mathrm{Inv}(\sigma)}$$

where $\mathrm{Inv}(\sigma) = |\{ i<j : \sigma(i) > \sigma(j) \}|$.

1 Mallows permutations

$\pi_{n,q}(\sigma) \propto q^{\mathrm{Inv}(\sigma)}$

From the definition, we have :

- When $q=0$, $\pi_{n,0}$ gives weight 1 to id
- When $q=1$, $\pi_{n,1}$ is the uniform distribution
- When $q\to\infty$, $\pi_{n,q}$ gives weight $\to 1$ to $r=(n,\dots,1)$

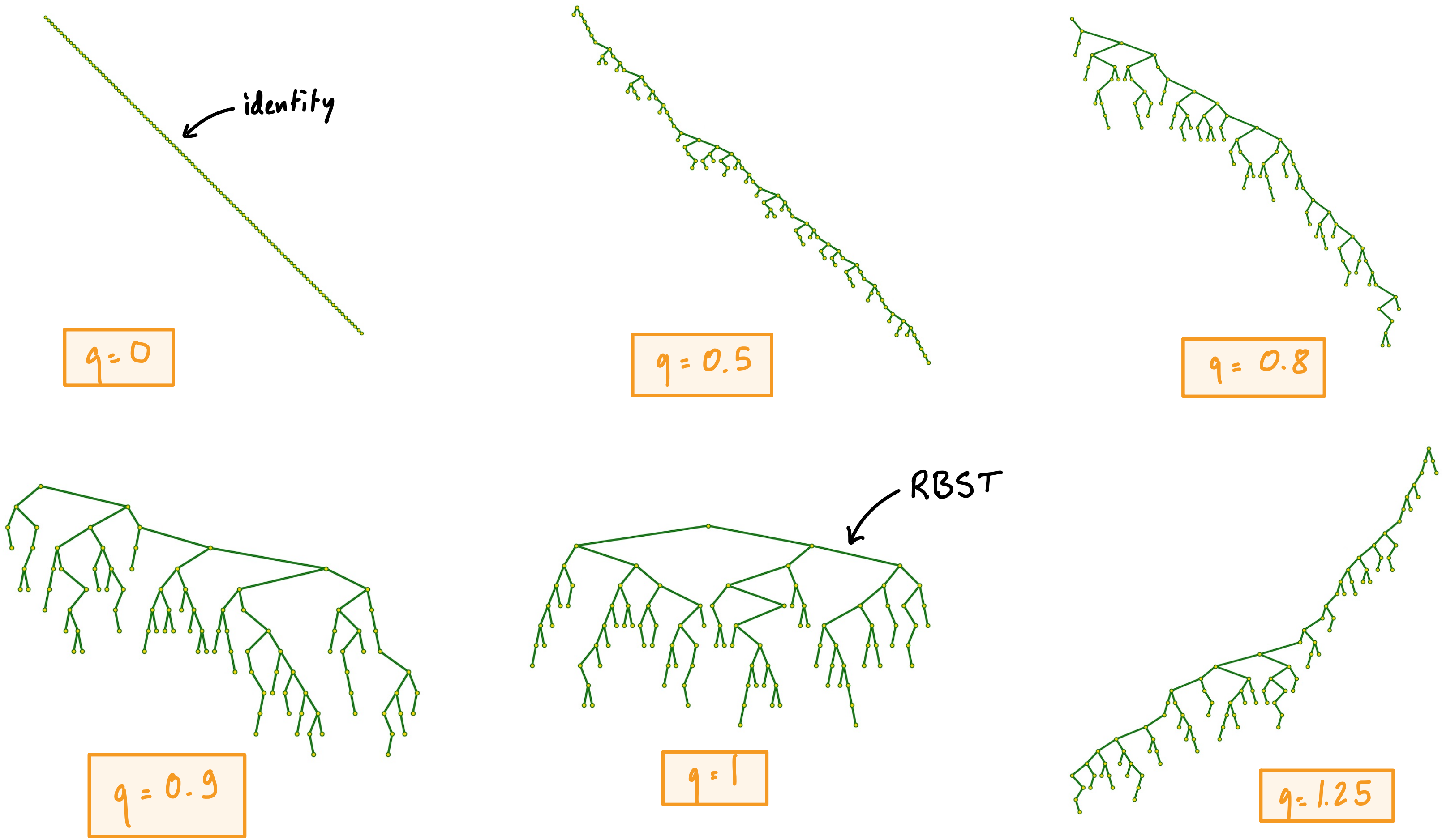
$\longrightarrow$ From $q=0$ to $q=1$, Mallows permutations go from ordered to random.

$\longrightarrow$ Past $q \geq 1$, a symmetry relates $\pi_{n,q}$ to $\pi_{n,1/q}$.

Examples of Mallows trees (n,q):

$\pi_{n,q}(\sigma) \propto q^{Inv(\sigma)}$

$n = 100$



2 Results on the height

Devroye's theorem

Let B_n be RBST with n nodes.
Then

$$\frac{\mathrm{ht}(B_n)}{c^* \log n} \xrightarrow{\mathbb{P}} 1 .$$

2 Results on the height

Our results

Let $T_{n,q}$ be Mallows tree with params n, q.
Then

$$\frac{ht(T_{n,q})}{n(1-q^*)+\log n} \xrightarrow{\mathbb{P}} 1,$$

where $q^* = \min(q, 1/q)$.

For $q \leq 1$, $q^* = q$.

2 Results on the height

When proving the previous result, we also proved that, if $q<1$ and $n(1-q)/\log n \to \infty$,

- $\dfrac{ht(T_{n,q})}{n(1-q)} \xrightarrow{a.s.} 1$

Central Limit Theorem

- $\dfrac{ht(T_{n,q}) - n(1-q) - c^* \log(1/q)}{\sqrt{n(1-q)q}} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$ if $nq \to \infty$

- $n-1-ht(T_{n,q}) \xrightarrow{\mathcal{L}} \text{Poisson}(\lambda)$ if $nq \to \lambda$

2 Results on the height

Takeaway from this:

- $\dfrac{\mathrm{ht}(T_{n,q})}{c^* \log n} \to 1$ when $n(1-q)/\log n \to 0$ includes RBST
- $\dfrac{\mathrm{ht}(T_{n,q})}{n(1-q)} \to 1$ when $q<1$ is fixed linear growth
- Between these two regimes, a combination of both factors.

2 Results on the height

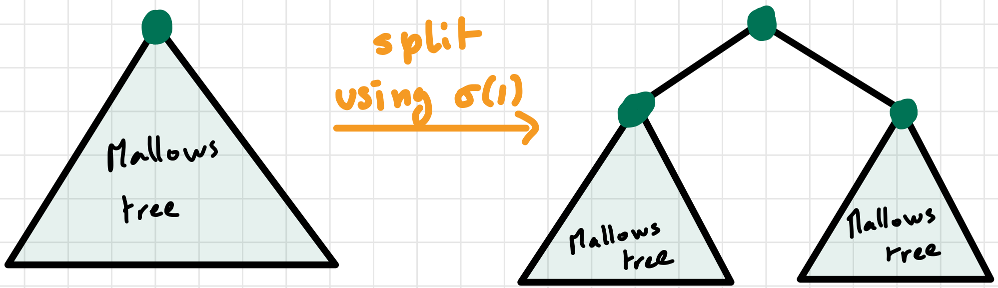
Overall, one formula summarizes all results:

$$\mathrm{ht}(T_{n,q}) \simeq \underbrace{n(1-q)}_{\text{linear}} + \underbrace{c^* \log\left(n \wedge \frac{1}{1-q}\right)}_{\text{logarithic}}$$

includes convergence & variations

3 Proof techniques

Mallows trees have properties similar to RBST and can be iterated via the root:



↳ Understanding the behaviour of $\sigma(1)$ suffices to characterize the trees.

3 Proof techniques

Recall that $\pi_{n,q}(\sigma) \propto q^{\mathrm{Inv}(\sigma)}$ and note that $\mathrm{Inv}(\sigma) = \left|\{ i<j : \sigma(i) > \sigma(j)\}\right|$

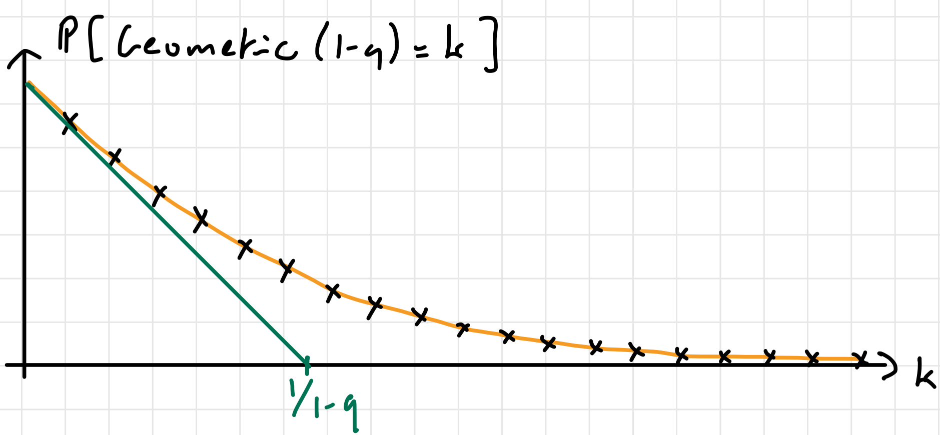
$$= \sigma(1) - 1 + \left|\{ 1<i<j : \sigma(i) > \sigma(j)\}\right| .$$

$$\longrightarrow \mathbb{P}[\sigma(1) = k] \propto q^{k}$$

$$\mathbb{P}[\sigma(1) = k] = \frac{q^{k-1}(1-q)}{1-q^n}$$

3 Proof techniques

$\sigma(1)$ is a Geometric$(1-q)$ conditioned to be in $\{1,2,\dots,n\}$



- If $n = O\left(\frac{1}{1-q}\right)$:

  $\sigma(1) \in_u^{*} \{1,2,\dots,n\}$.

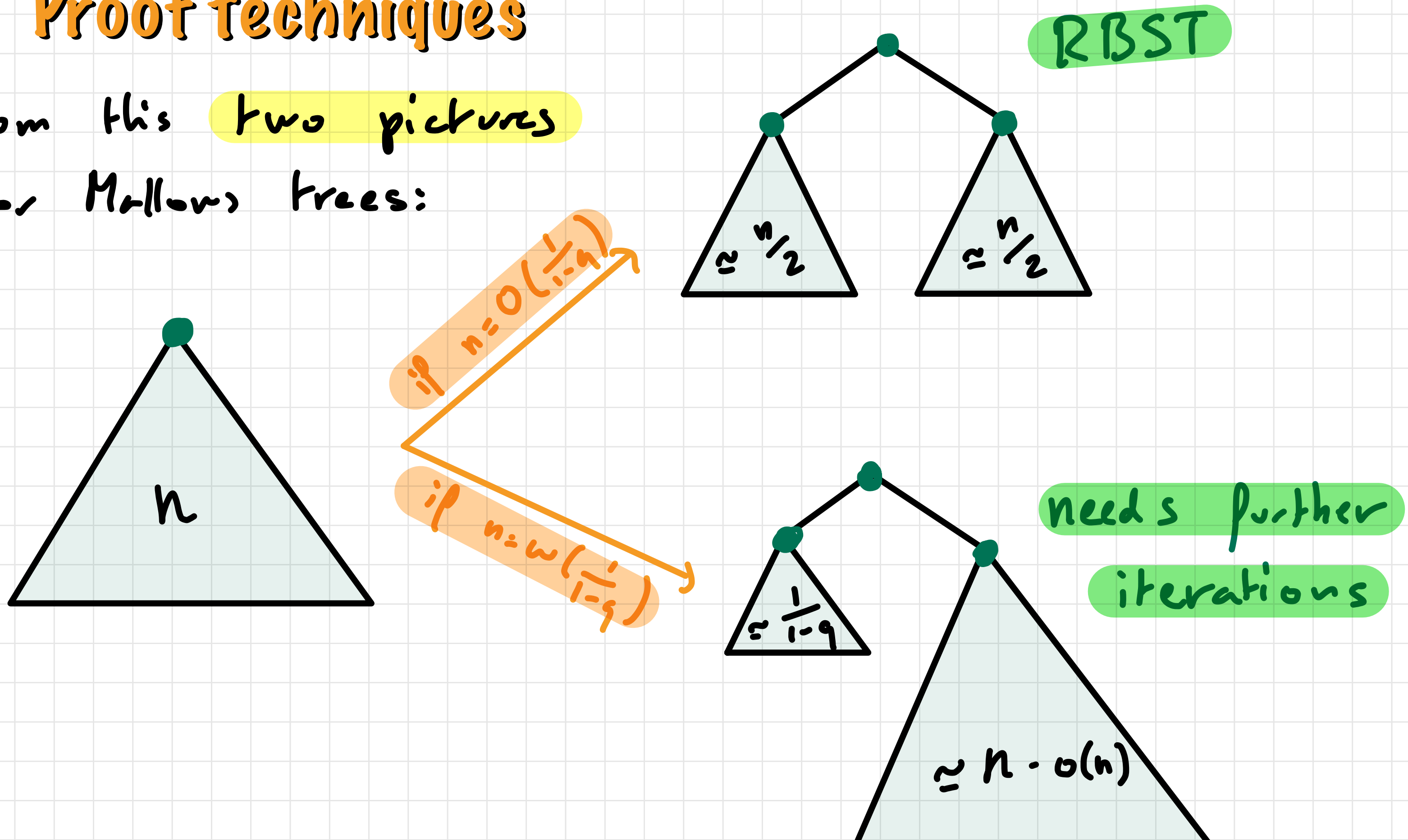
- If $n = \omega\left(\frac{1}{1-q}\right)$:

  $\sigma(1) \simeq \frac{1}{1-q} << n$

* it is uniform up to a constant

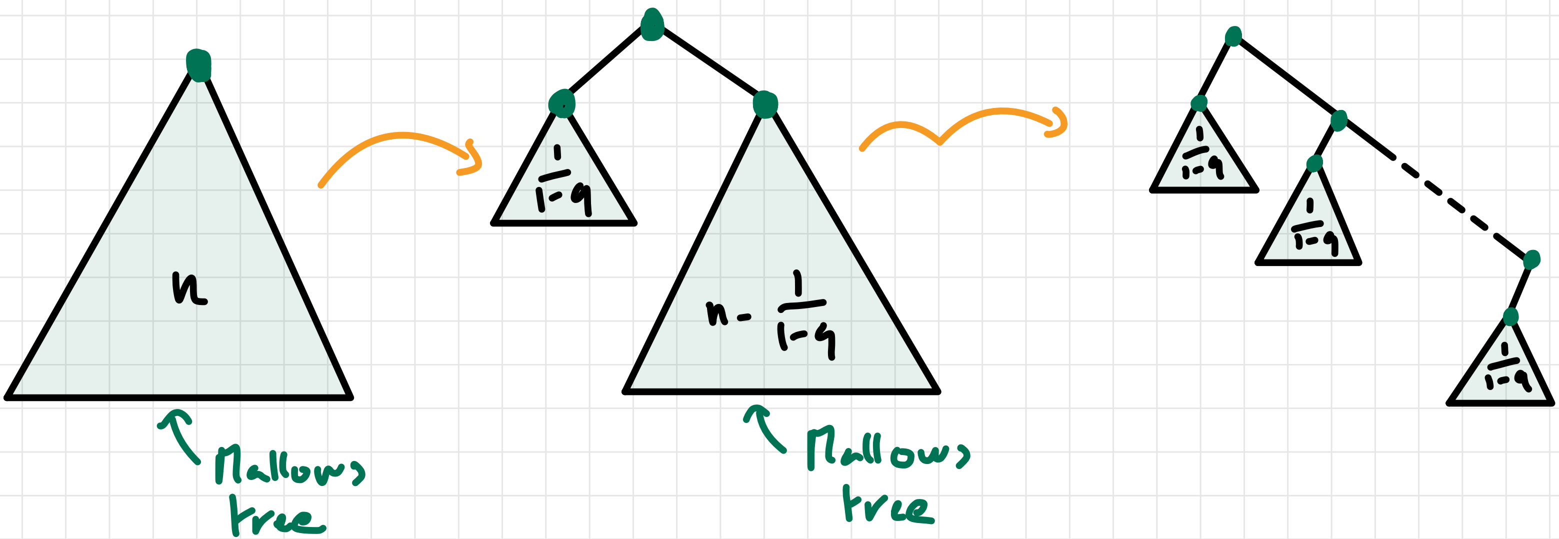
3 Proof techniques

From this two pictures
for Mallows trees:



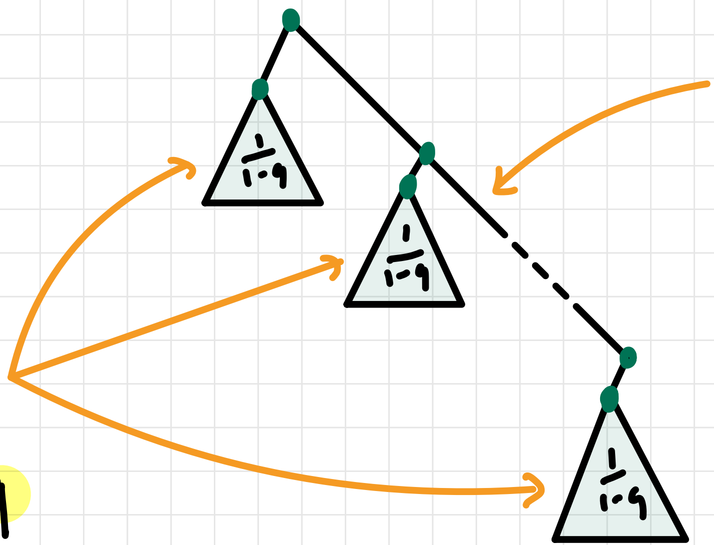
3 Proof techniques

For the second case, repeating the iteration process leads to:



3 Proof techniques

Consider now the last representation:



tree with n nodes and left subtrees of size $\frac{1}{1-q}$

↳ $n(1-q)$ such left subtrees.

Mallows trees of size $\frac{1}{1-q}$ and param q

↳ height $\approx c^* \log\left(\frac{1}{1-q}\right)$.

Proof techniques

In summary, we have:

- $\mathrm{ht}(T_{n,q}) \simeq c^* \log n$ if $n = o\left(\frac{1}{1-q}\right)$
- $\mathrm{ht}(T_{n,q}) \simeq n(1-q) + c^* \log\left(\frac{1}{1-q}\right)$ if $n = \omega\left(\frac{1}{1-q}\right)$

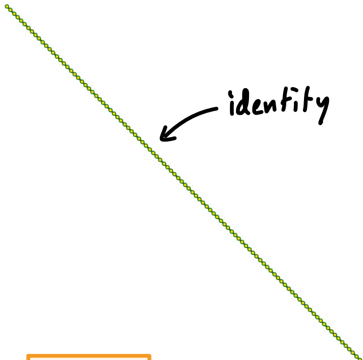
$$\longrightarrow \mathrm{ht}(T_{n,q}) \simeq \underbrace{n(1-q)}_{\text{linear}} + \underbrace{c^* \log\left(n \wedge \frac{1}{1-q}\right)}_{\text{logarithmic}}$$

Examples of Mallows trees (n,q):

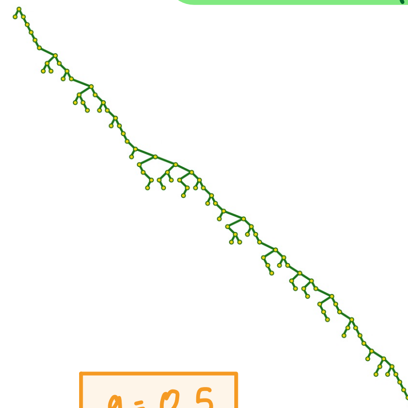
$\text{ht} \approx c^{\circ} \log n$ if $n = O\left(\frac{1}{1-q}\right)$

$\text{ht} = n(1-q) + c' \log\left(\frac{1}{1-q}\right)$ if $n = \omega\left(\frac{1}{1-q}\right)$

$n = 100$



$q = 0$



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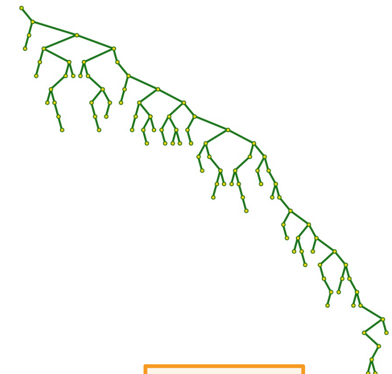

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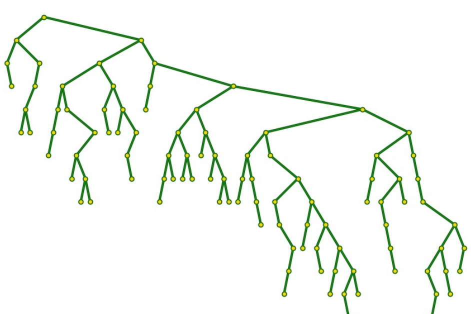

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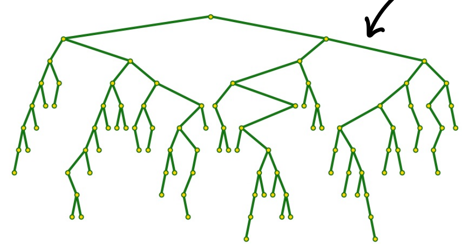

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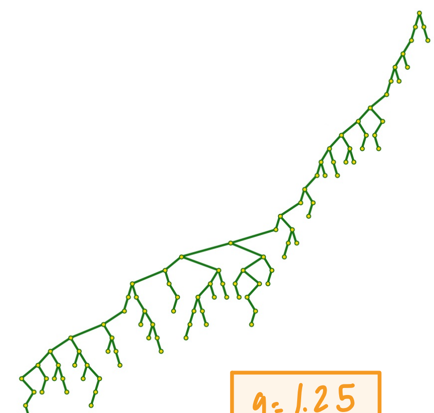

$q = 1.25$

Thank you!