The permuton tree

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joint work with Victor Dubach
and Valentin Féray

TU/e

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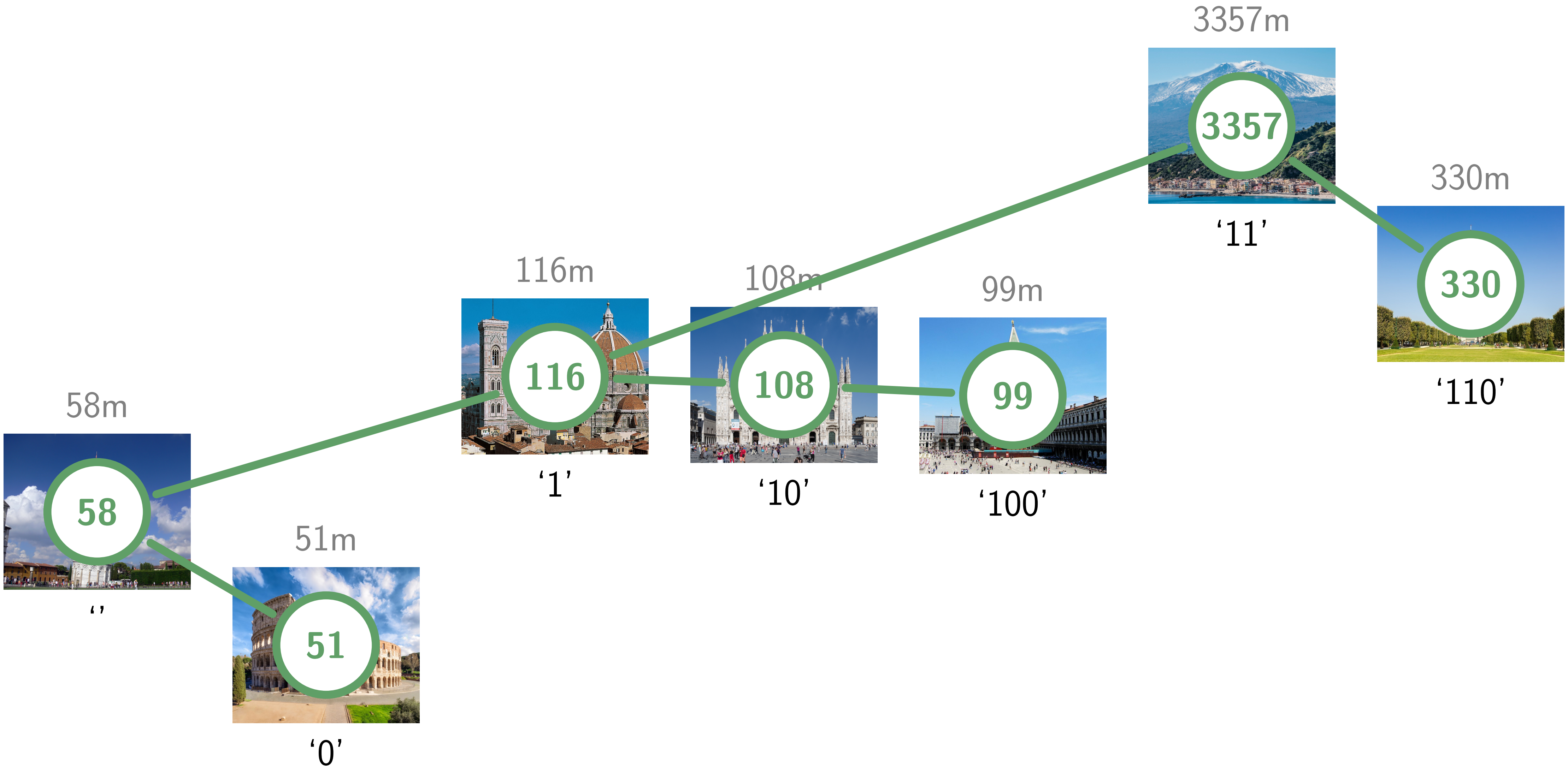
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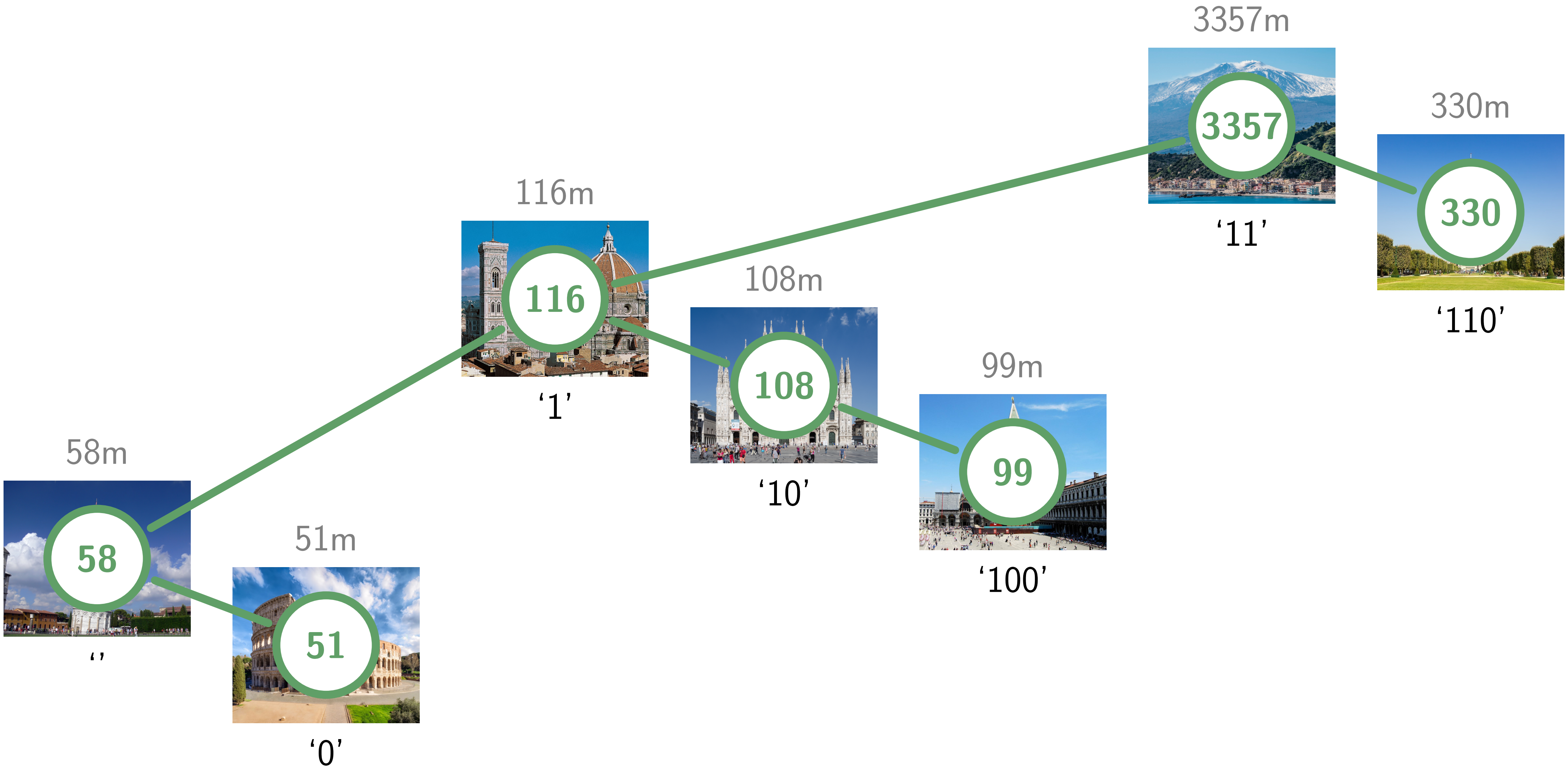
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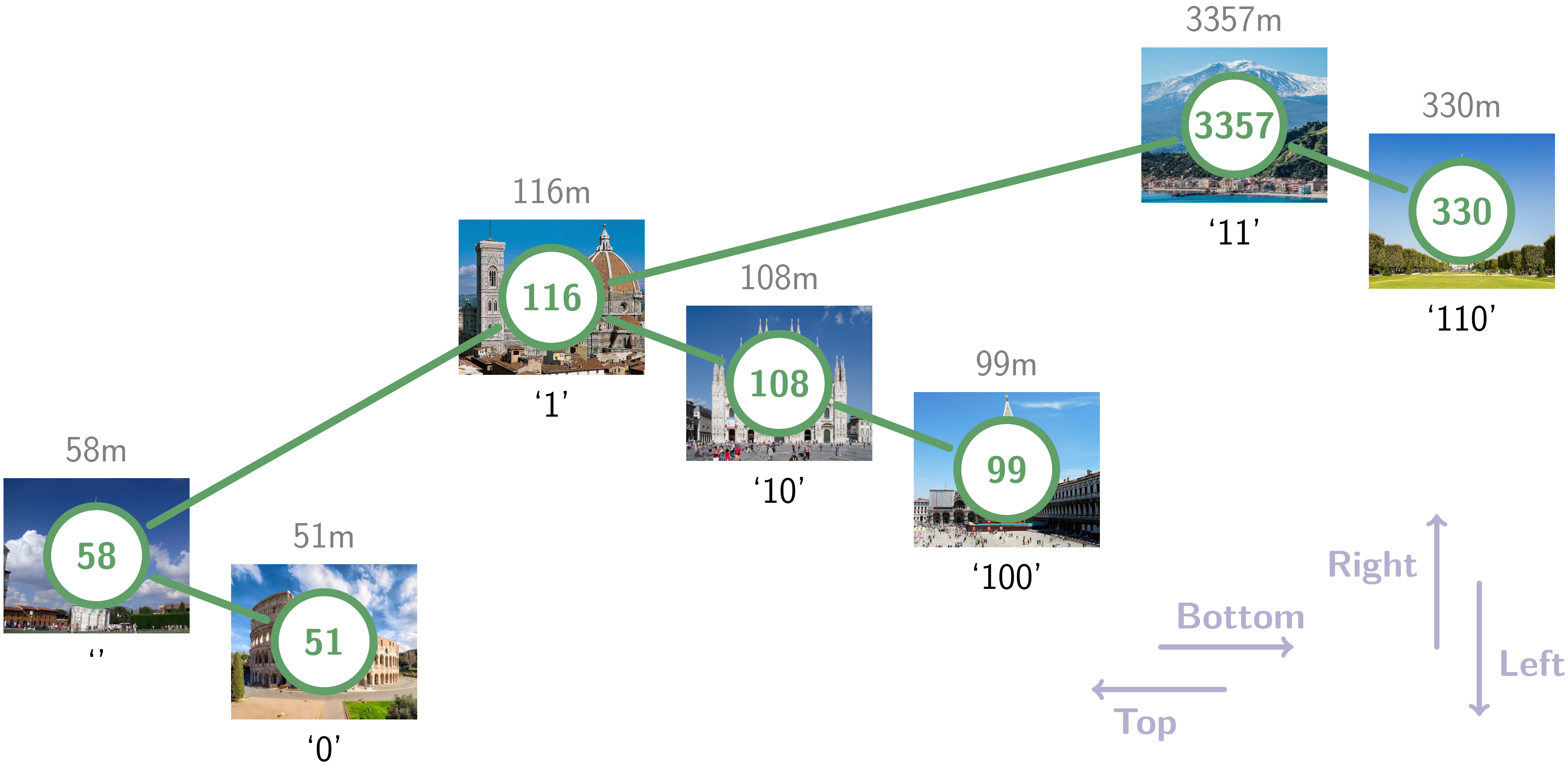

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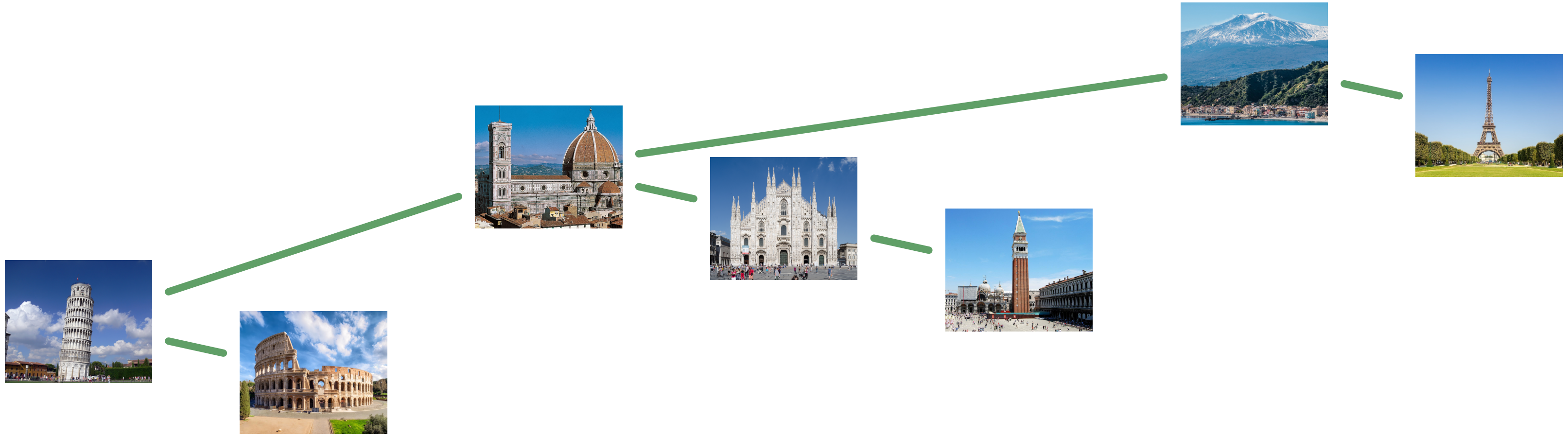

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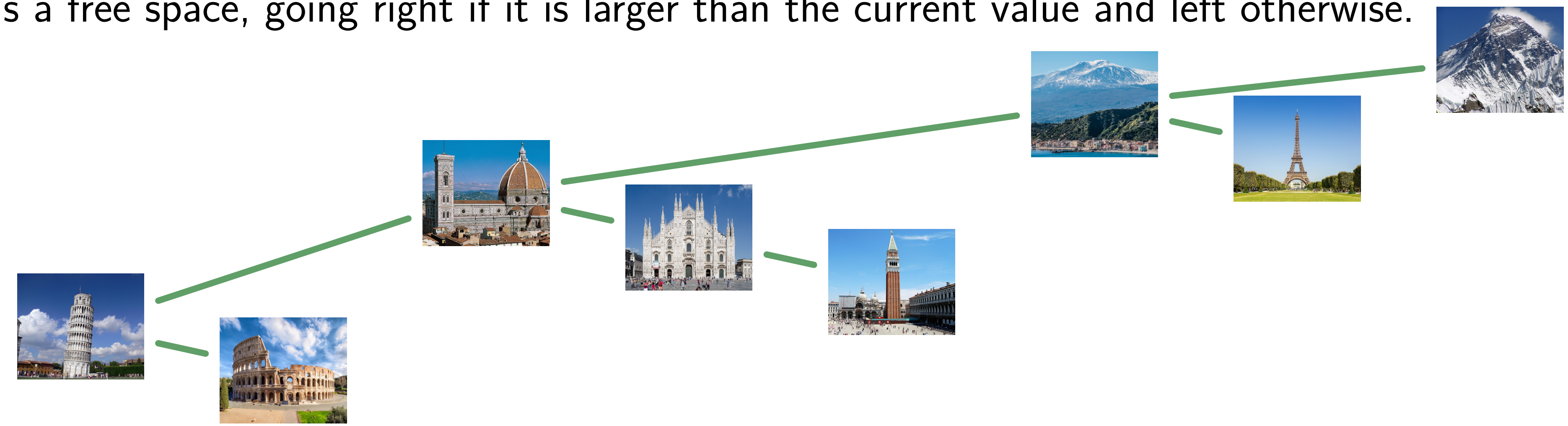

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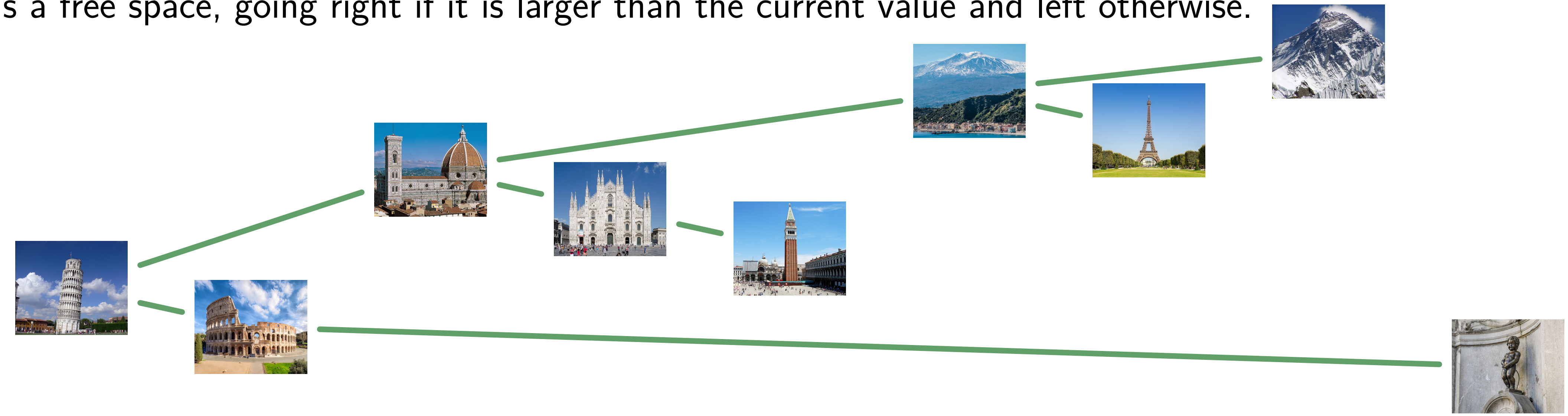

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→ Seen as storing devices, we are often interested in the "worst case retrieval time", corresponding to the height of the binary search tree.

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Devroye (1986)

Let H_n be the height of a RBST of size n. Then there exists a constant $c^* \simeq 4.311\ldots$ such that

$$\frac{H_n}{c^* \log n} \longrightarrow 1\,.$$

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→ We call such a distribution a *permuton*.

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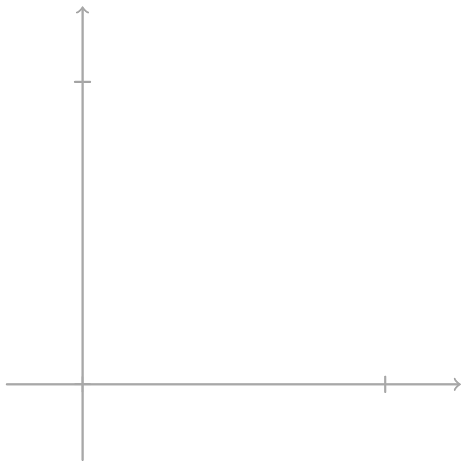

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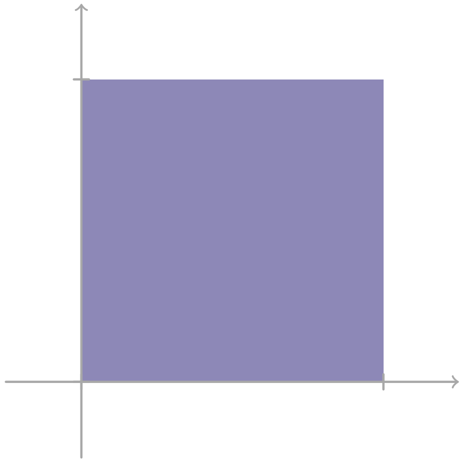

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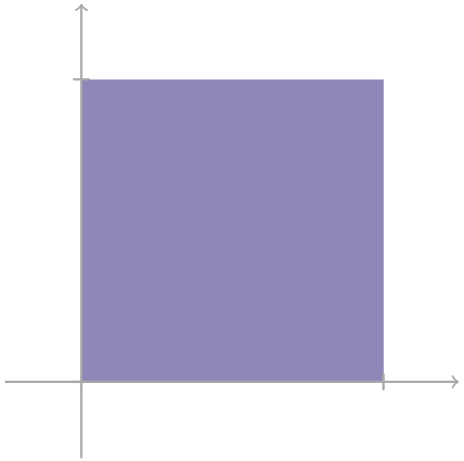

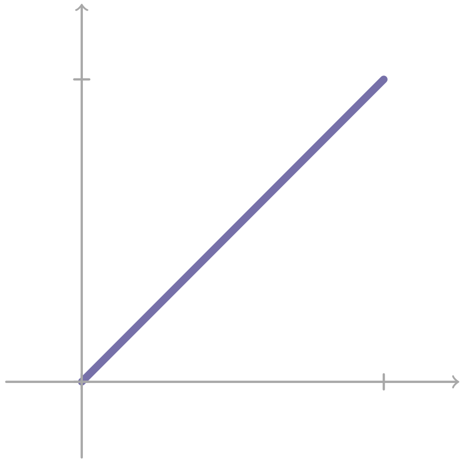

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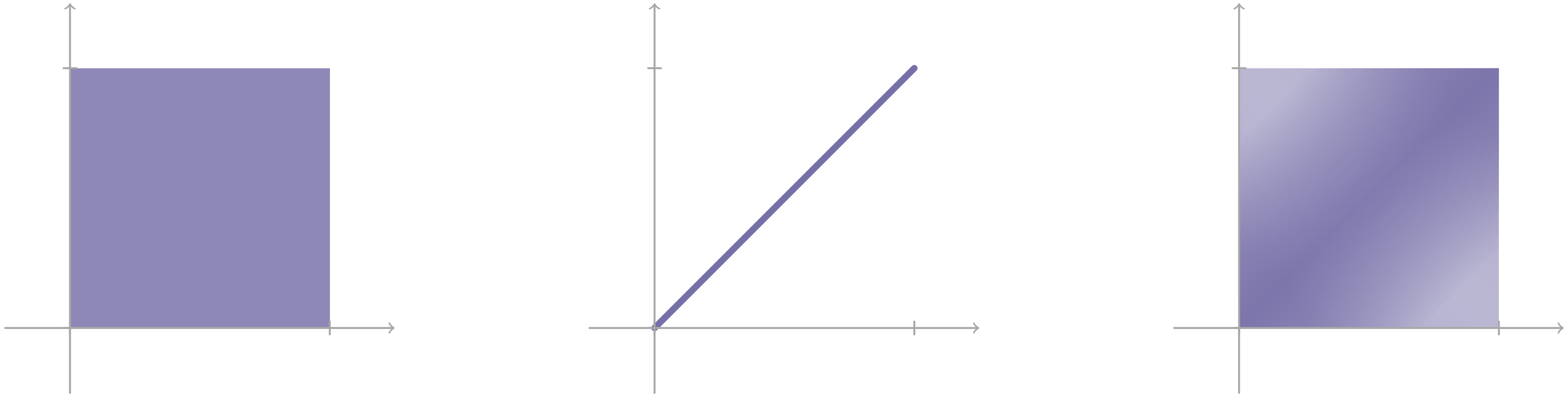

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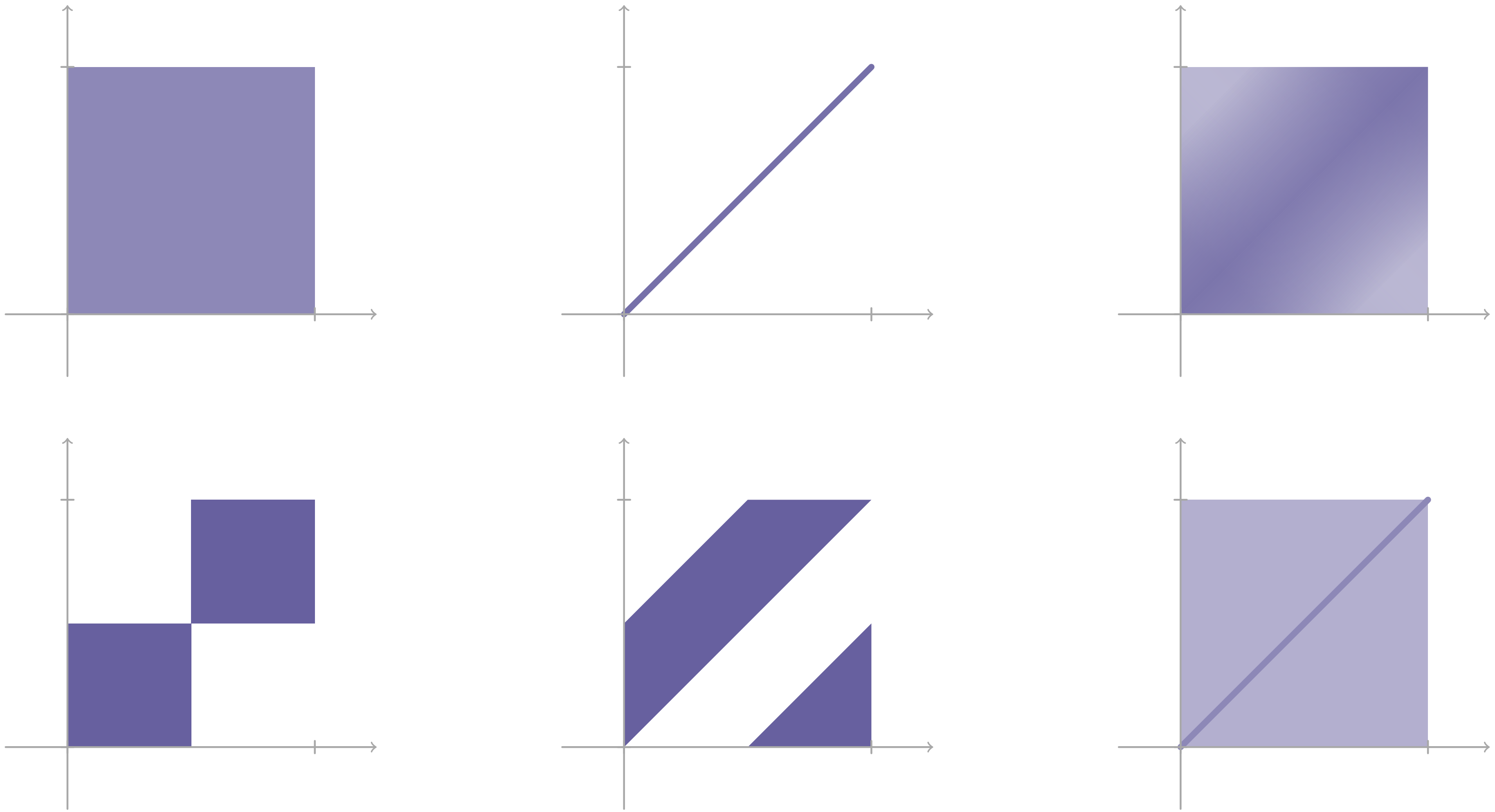

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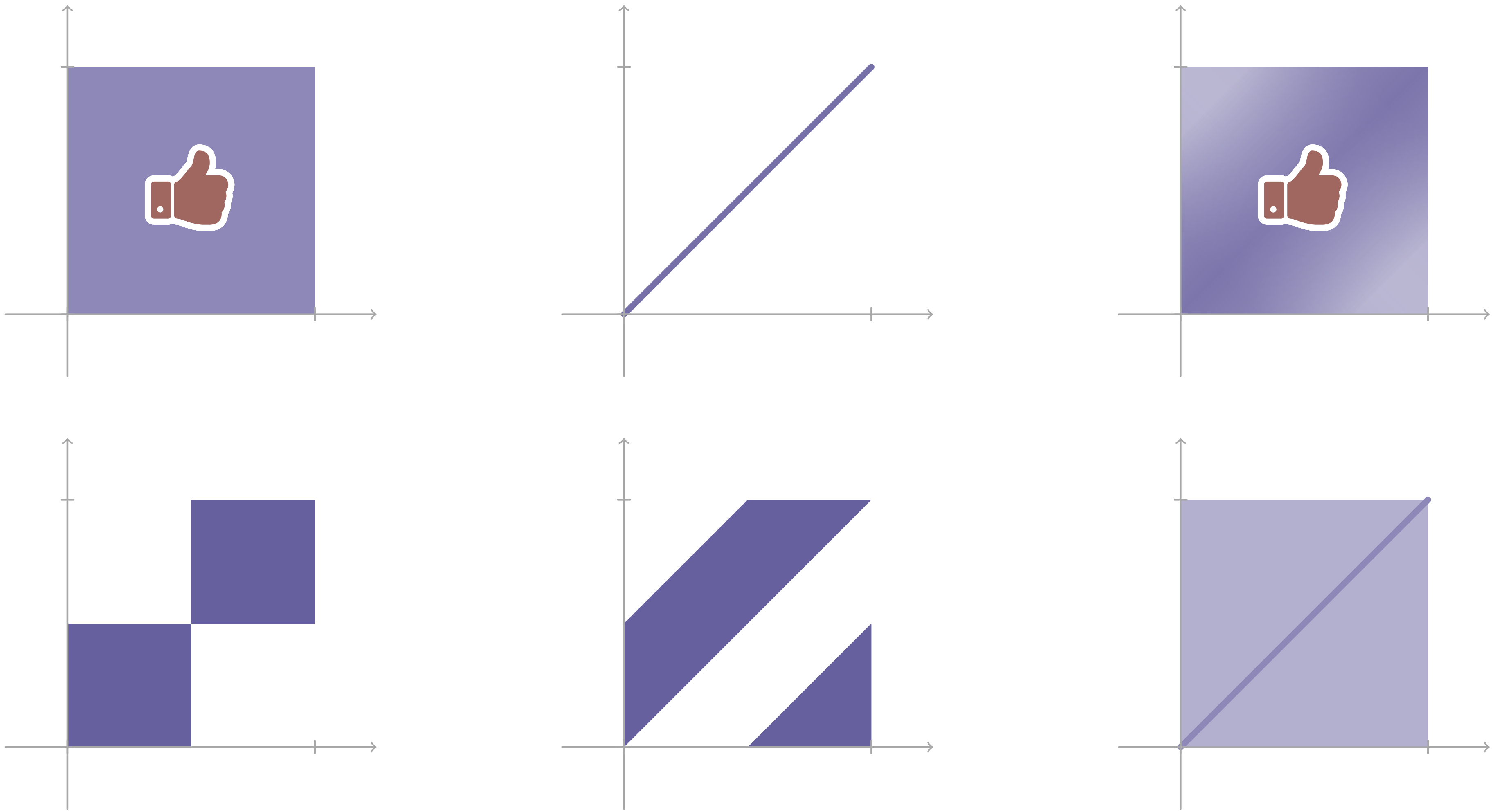

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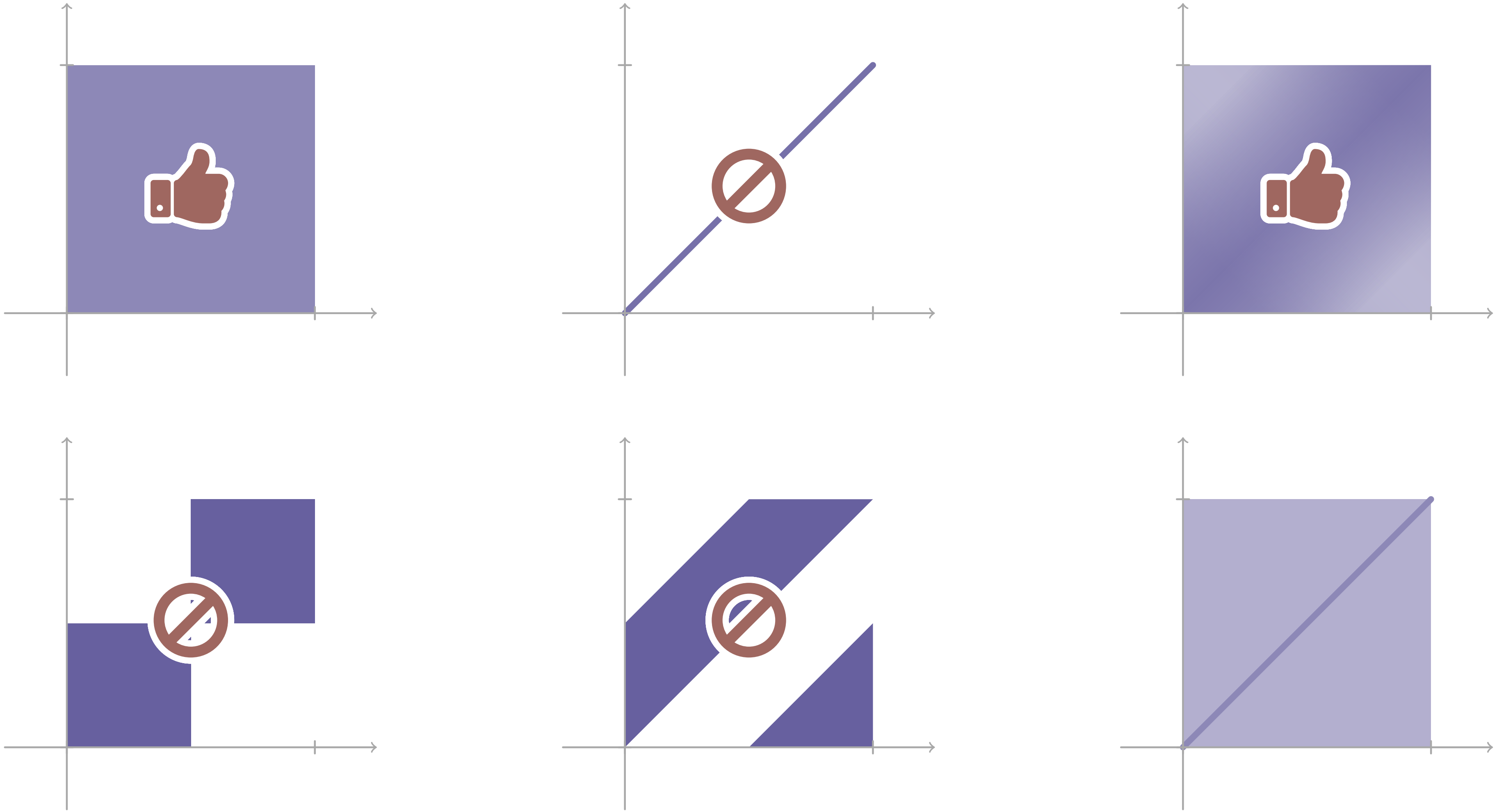

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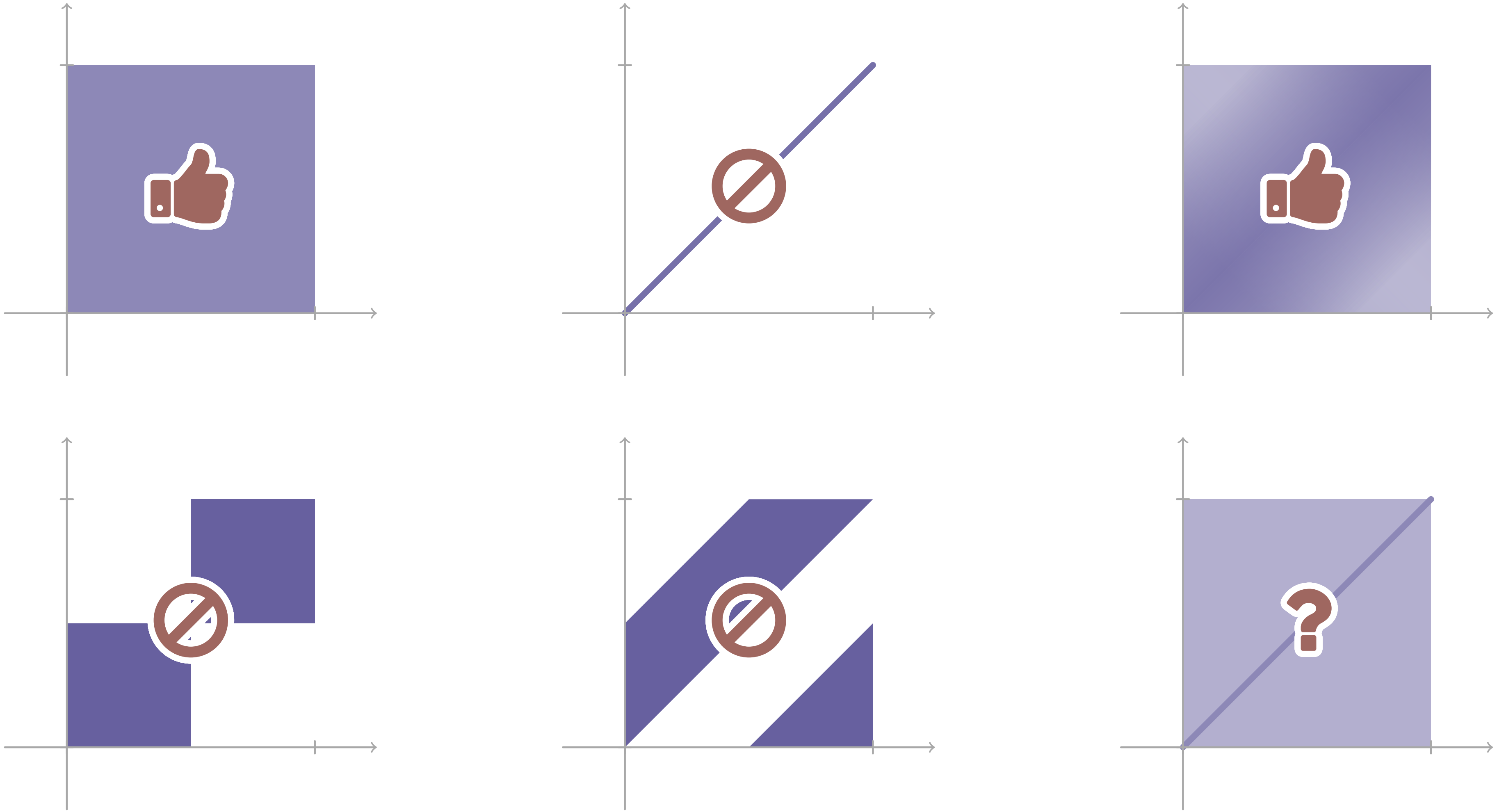

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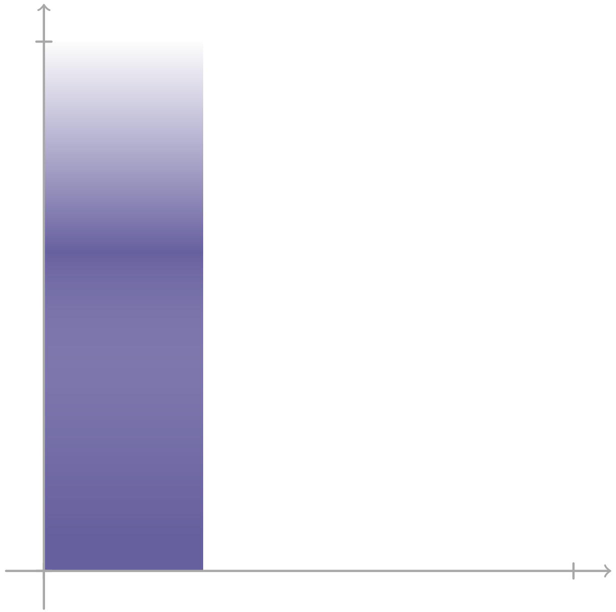

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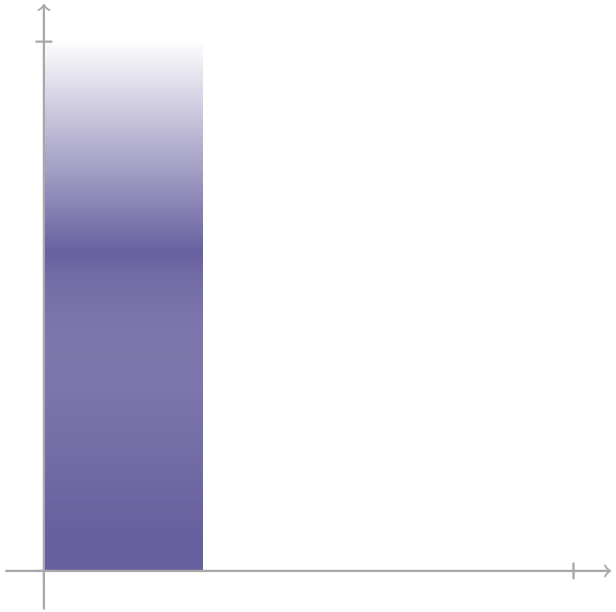

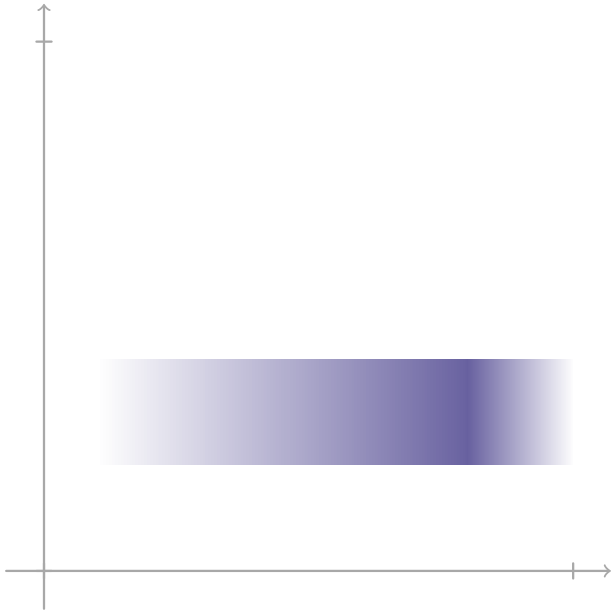

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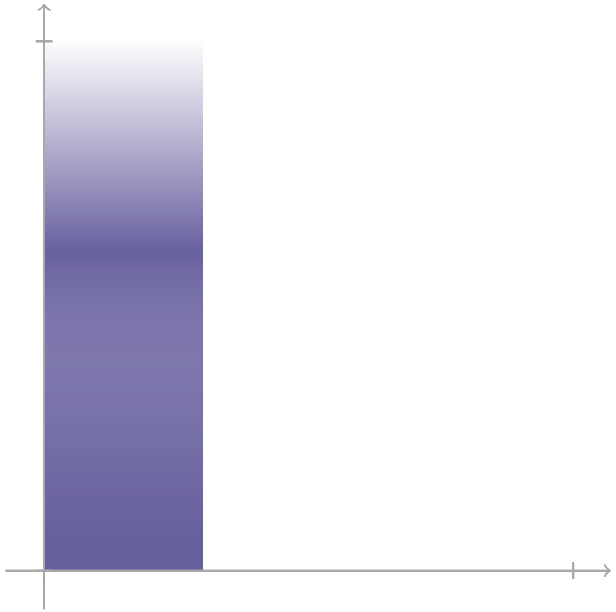

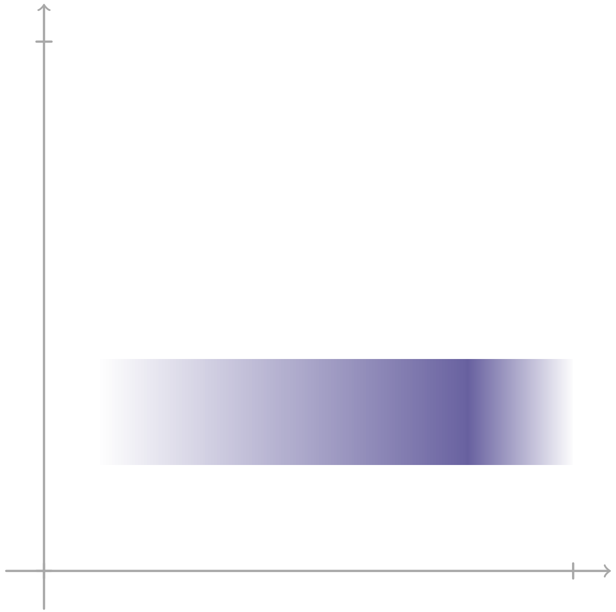

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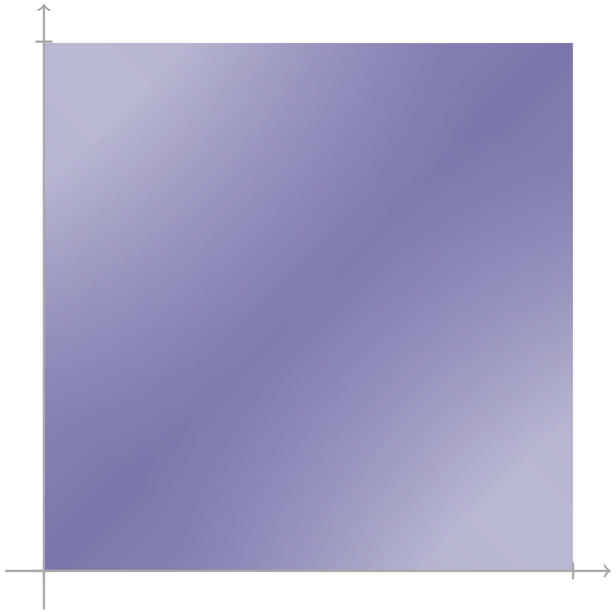

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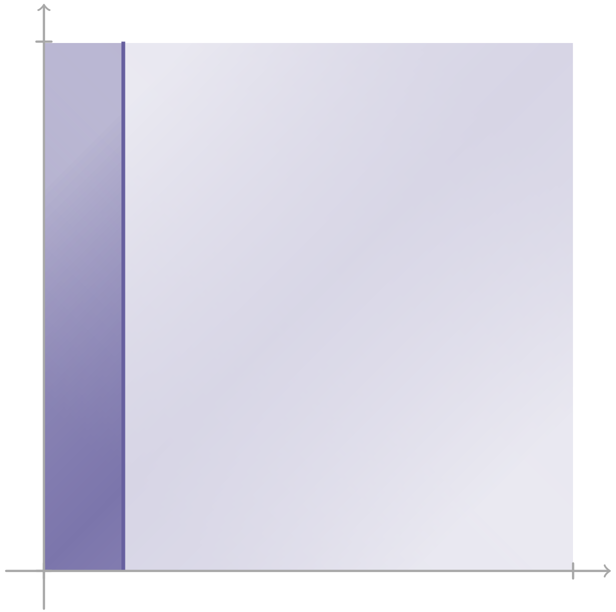

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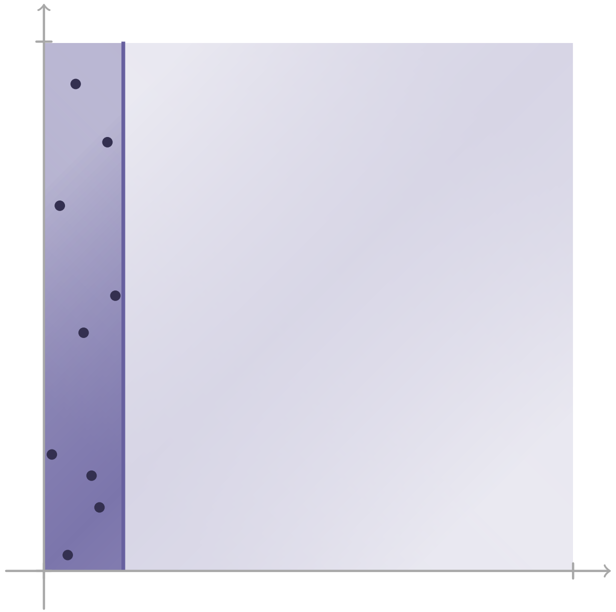

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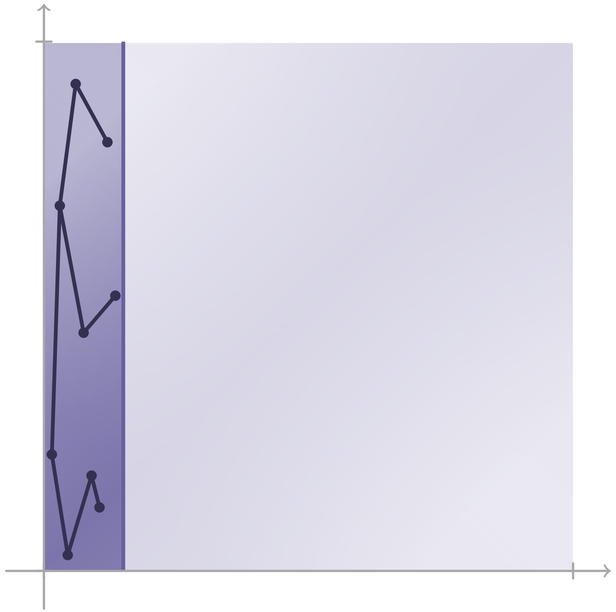

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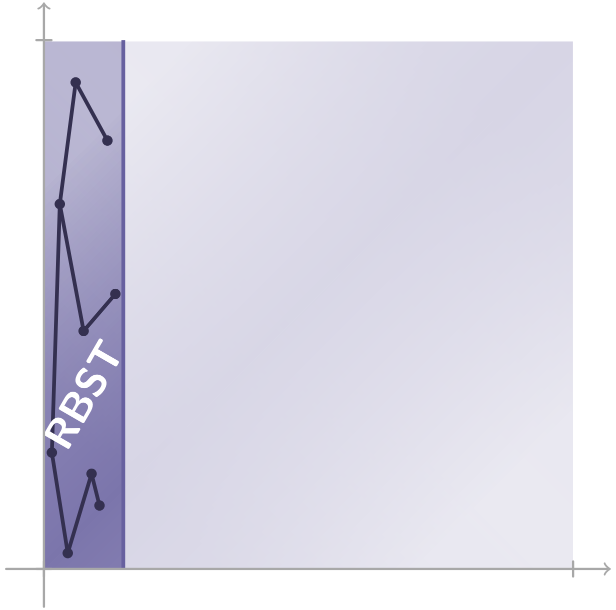

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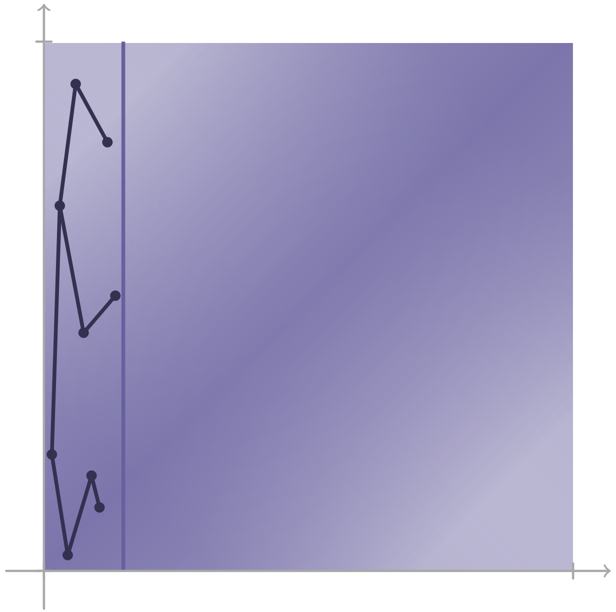

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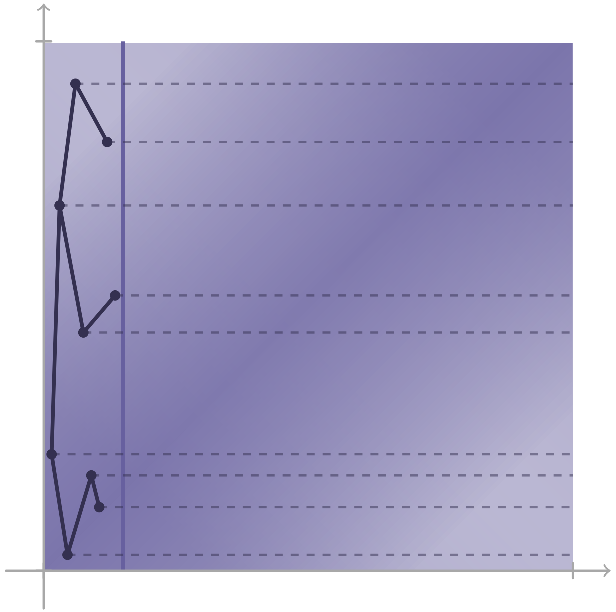

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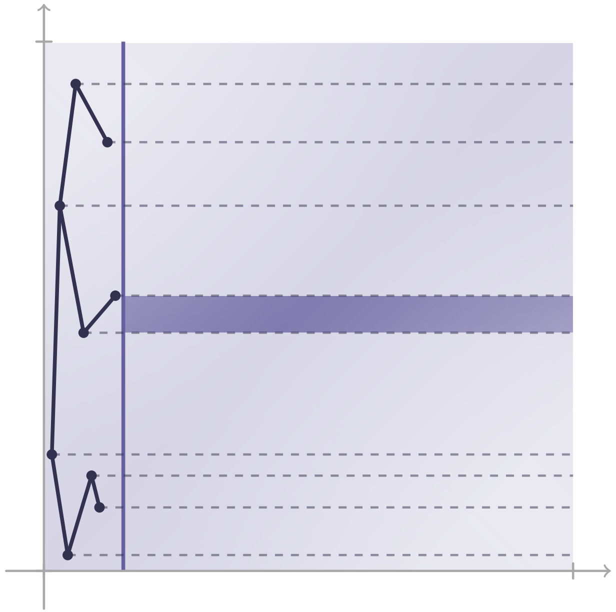

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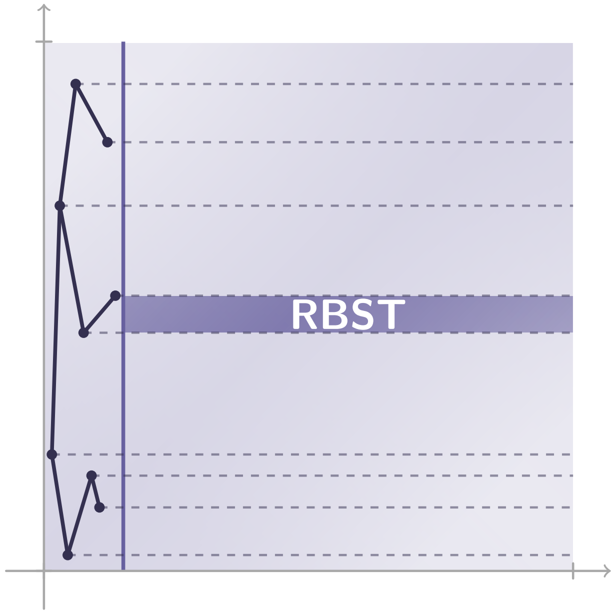

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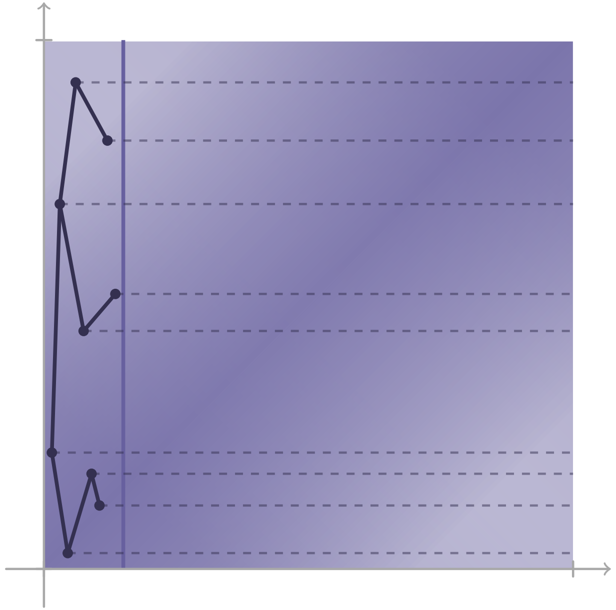

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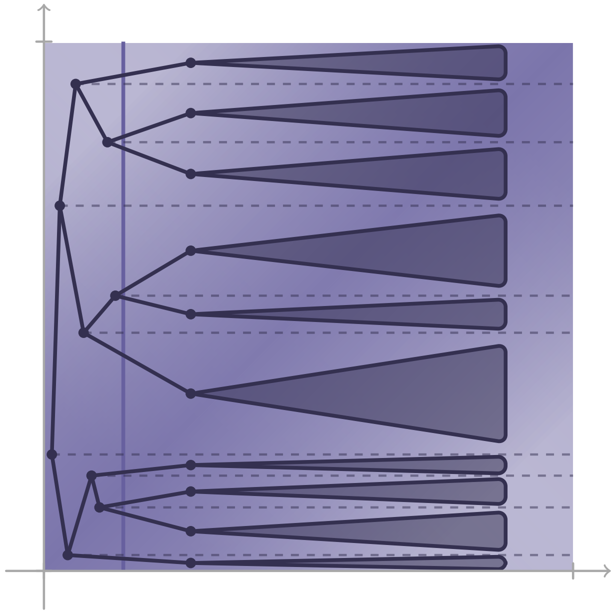

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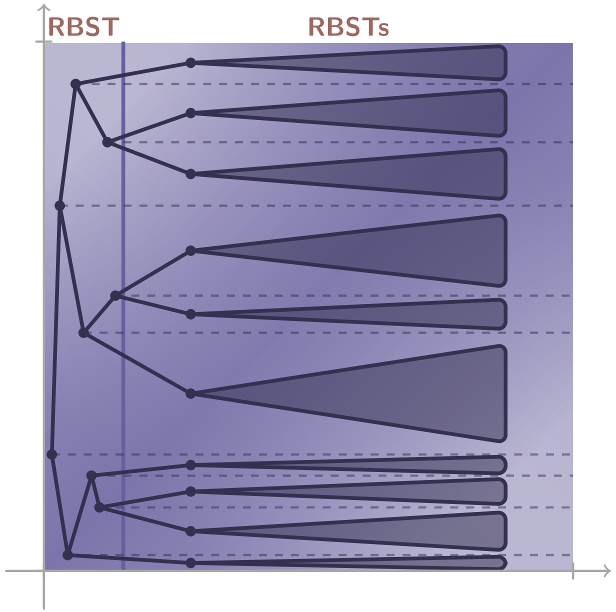

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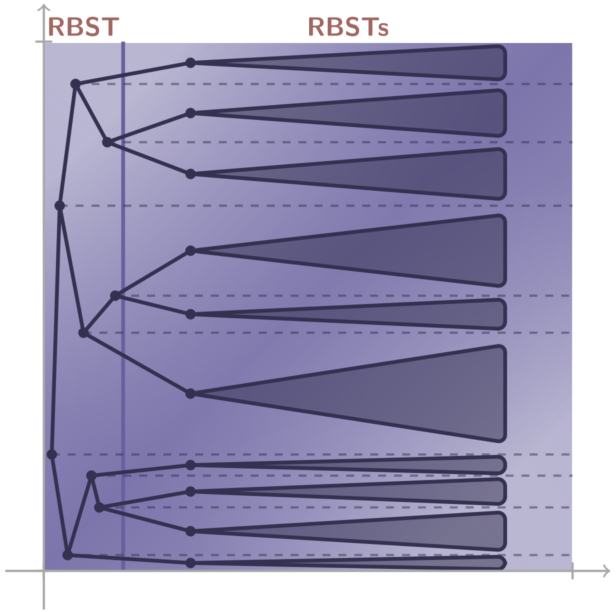

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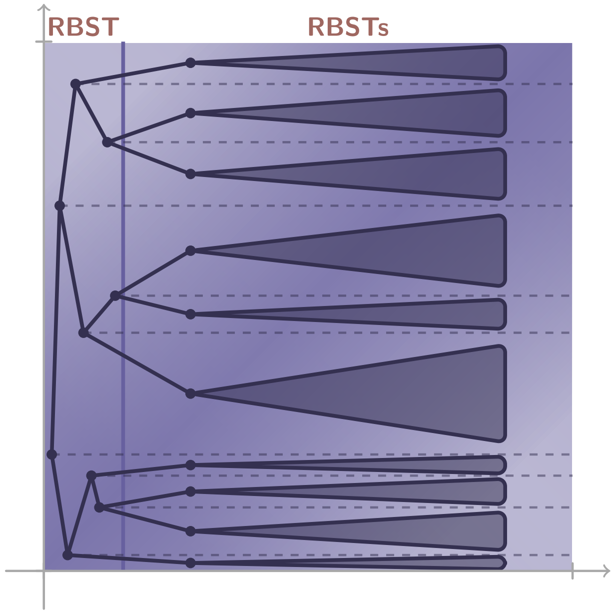

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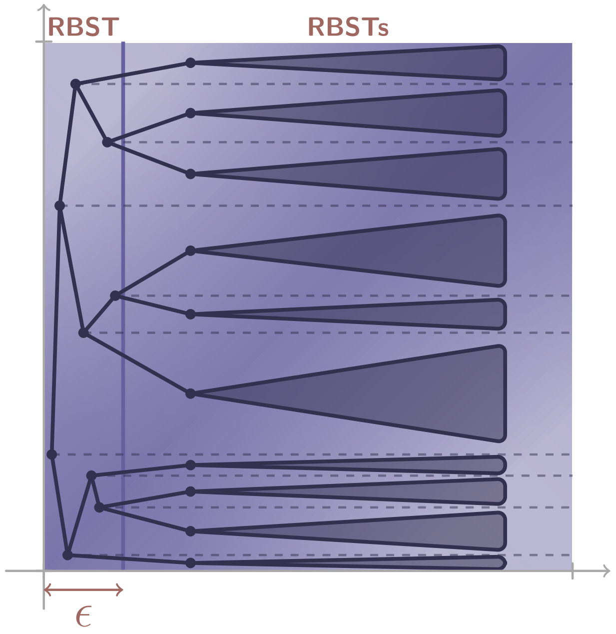

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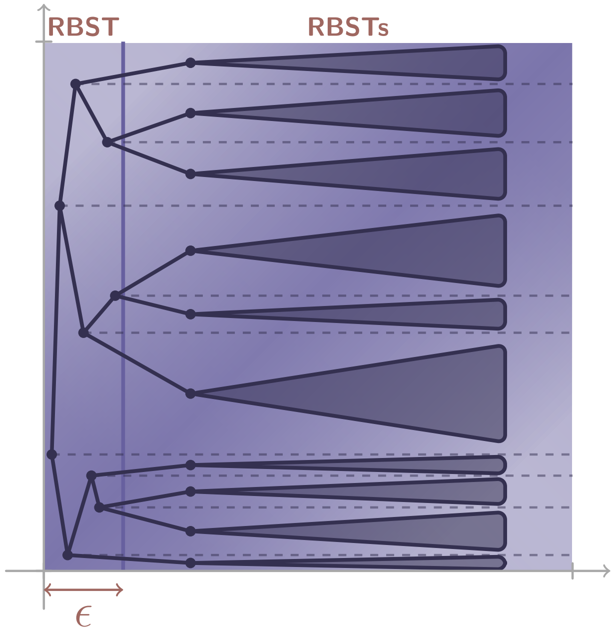

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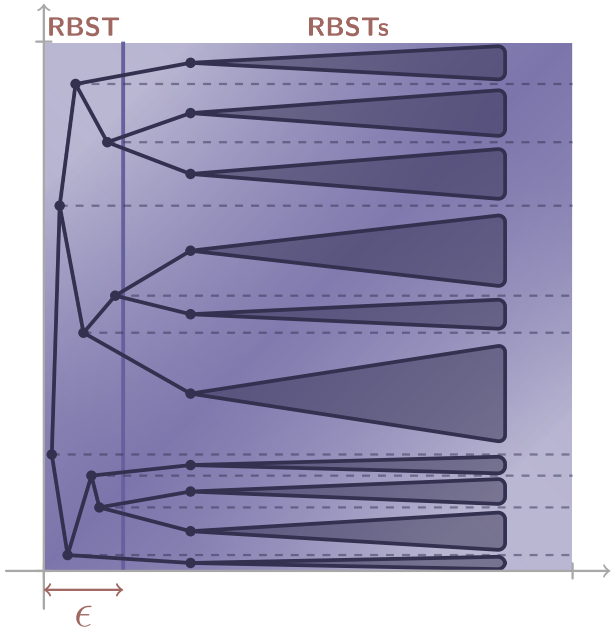

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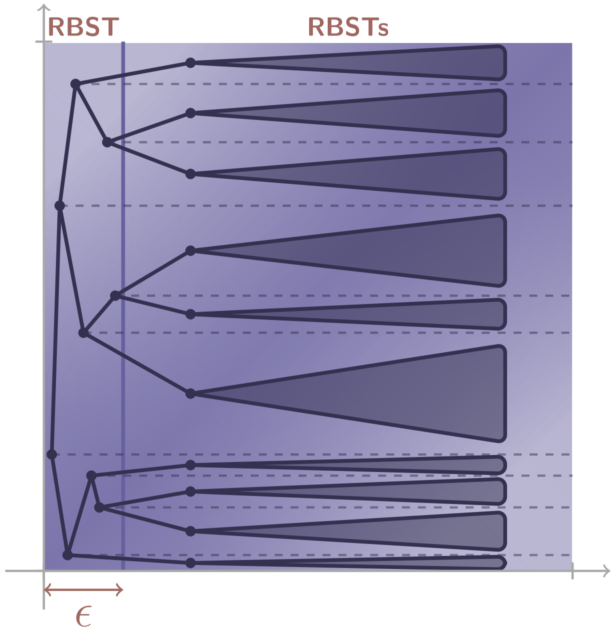

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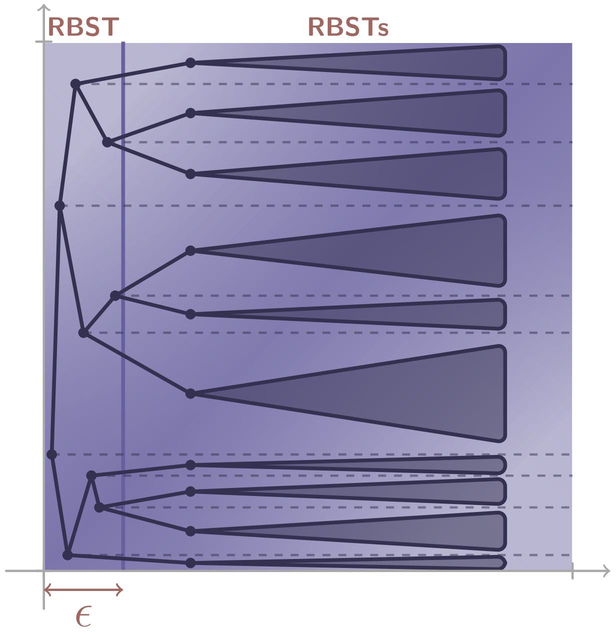

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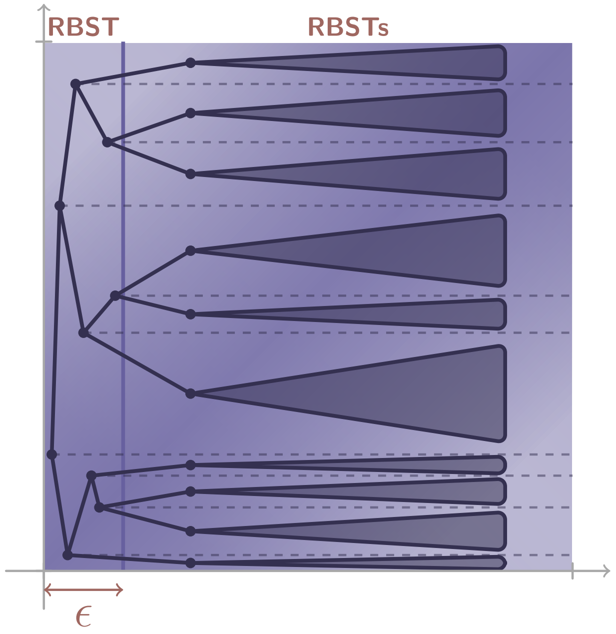

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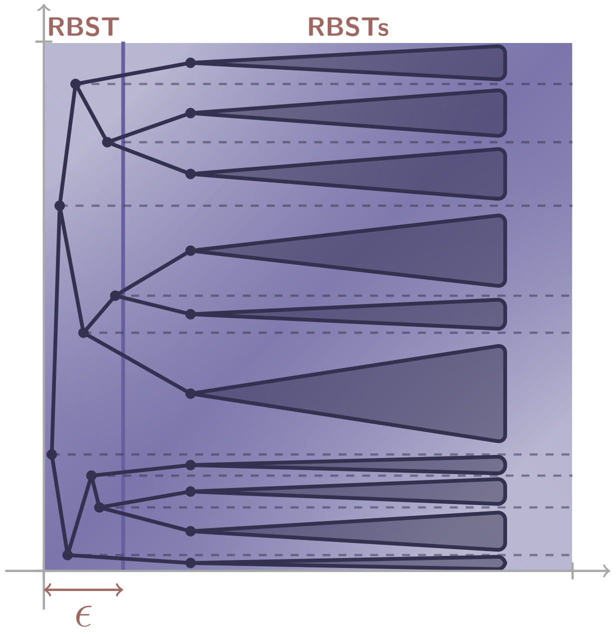

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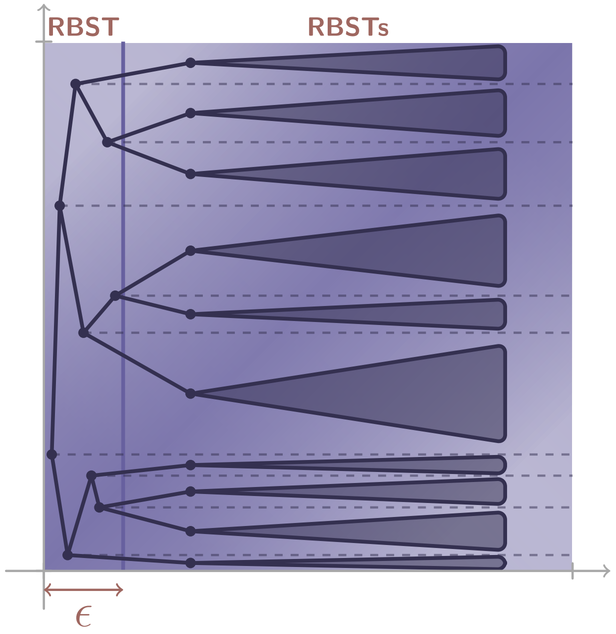

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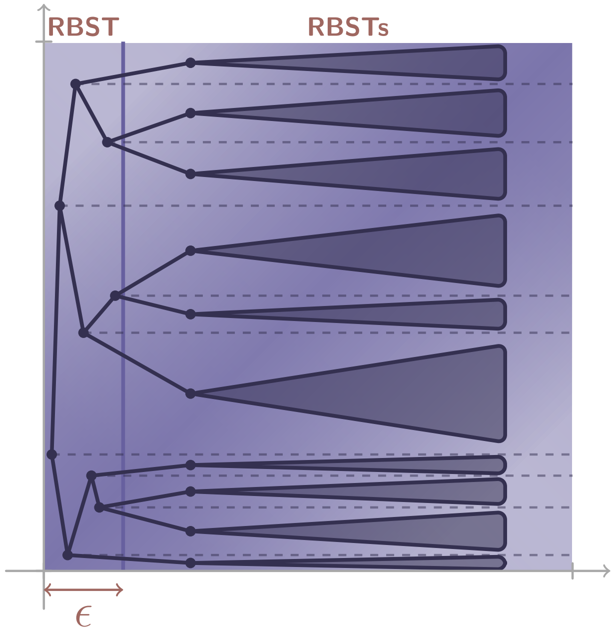
⚠ While the average width of a band is $1/\epsilon n$, the maximal width is actually of order $\log n/n$.



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→ We will now see why these conditions are necessary.

References

- Corsini, B., Dubach, V., & Féray, V. (2024). **Binary search trees of permuton samples.** *arXiv preprint arXiv:2403.03151.*
- Devroye, L. (1986). **A note on the height of binary search trees.** *Journal of the ACM (JACM), vol. 33, no 3, p. 489-498.*

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The conditions

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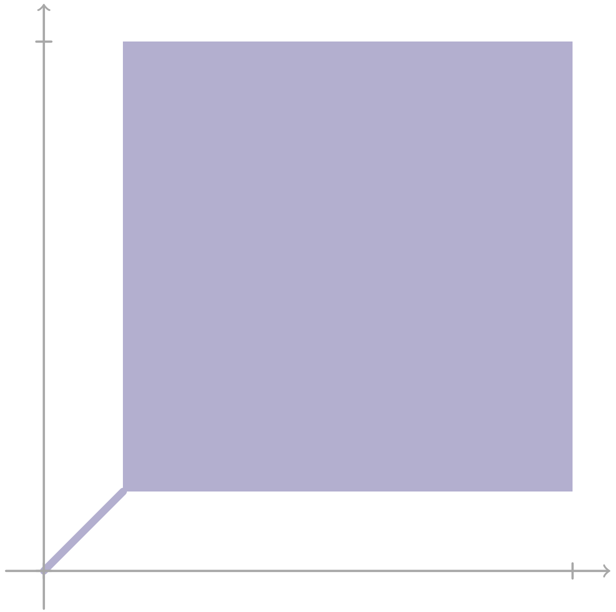

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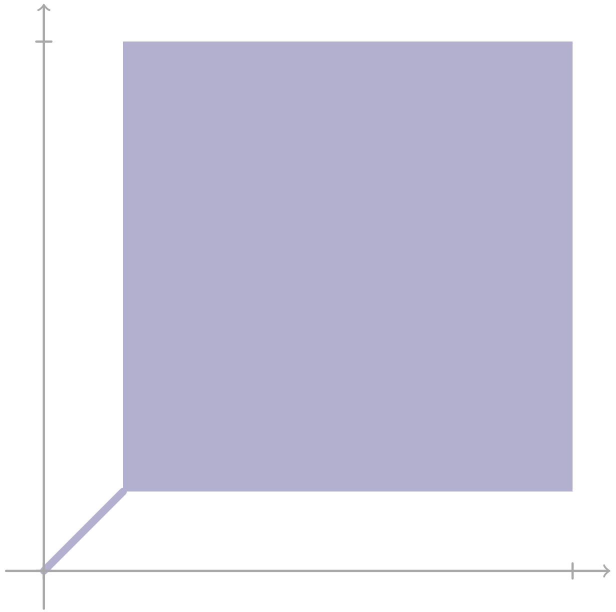

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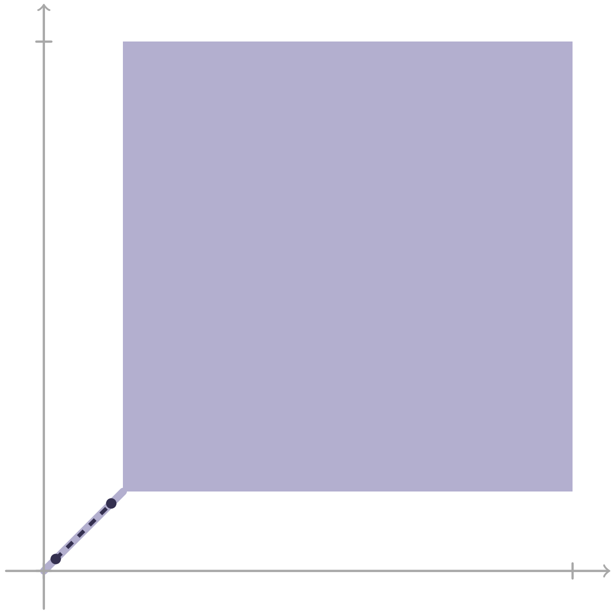

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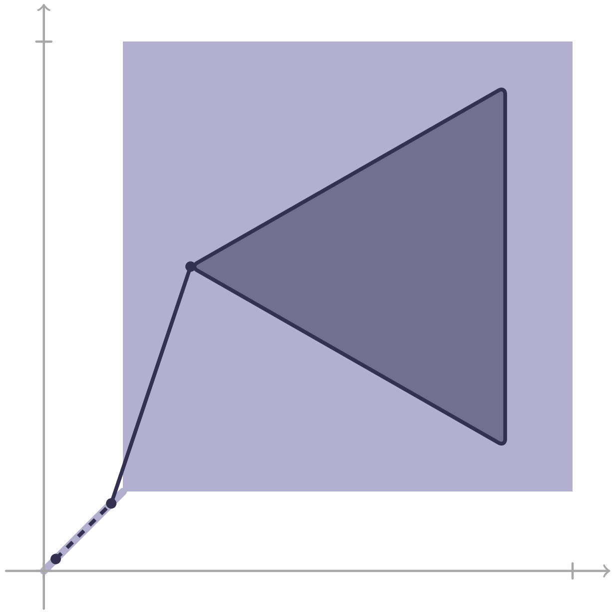

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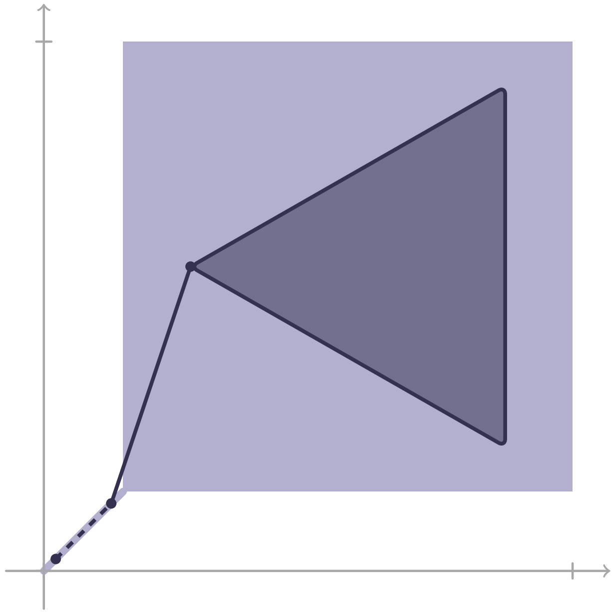

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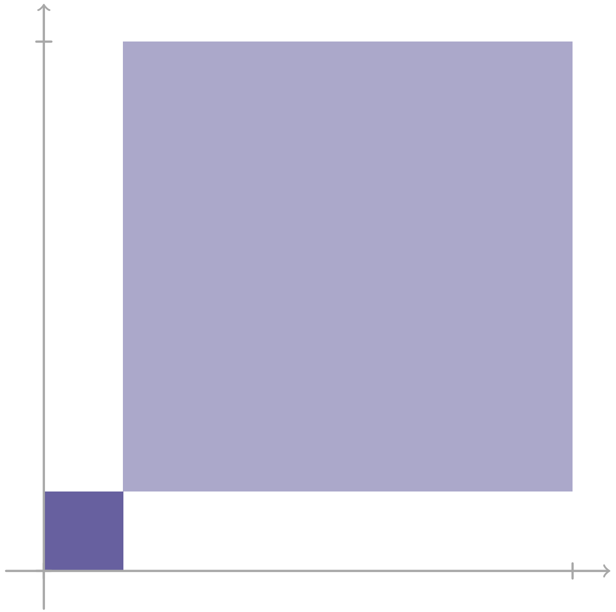

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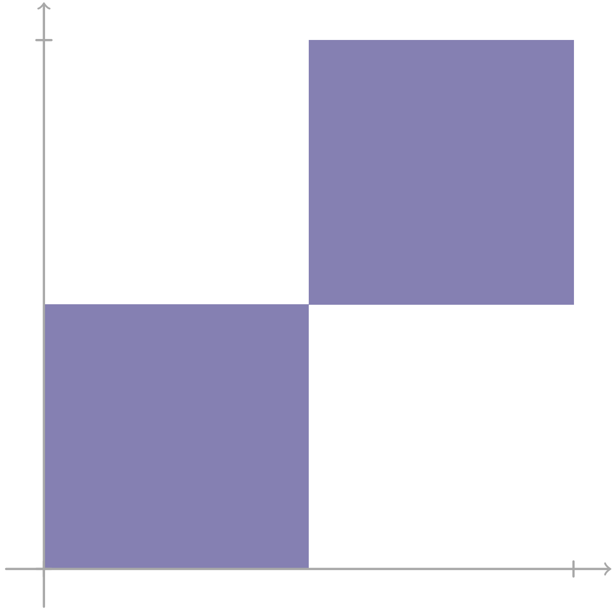

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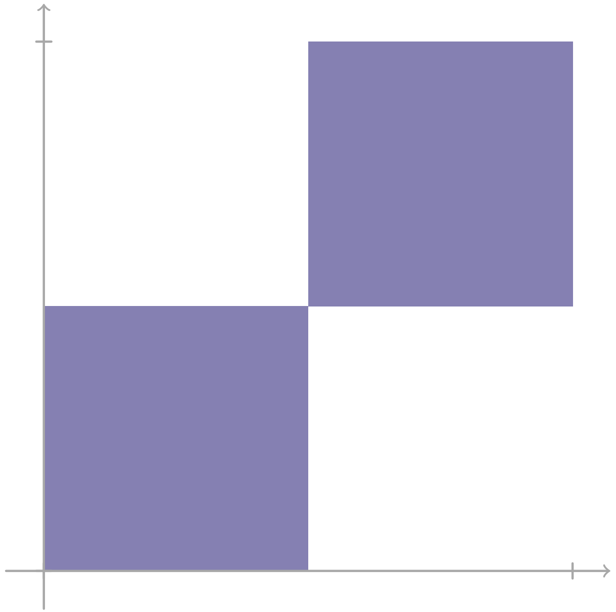

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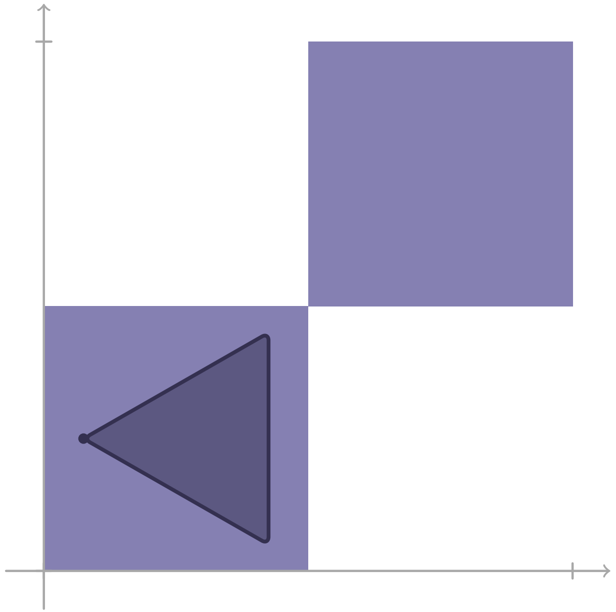

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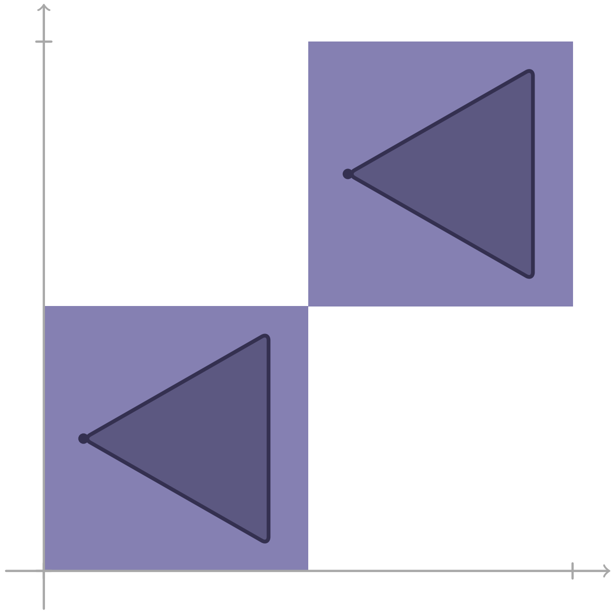

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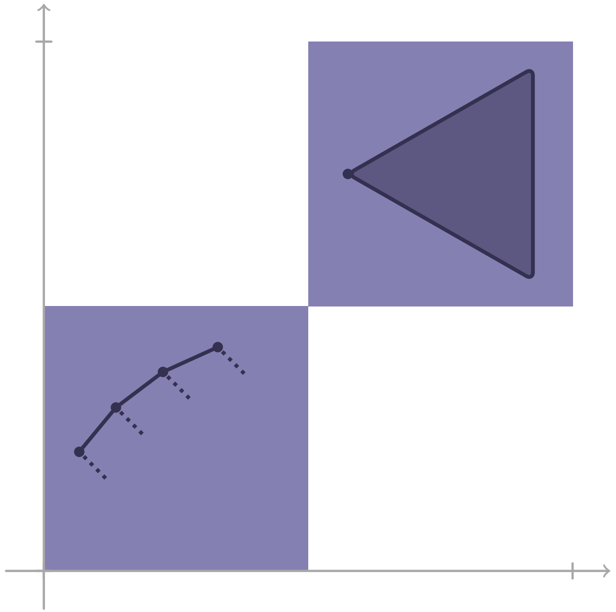

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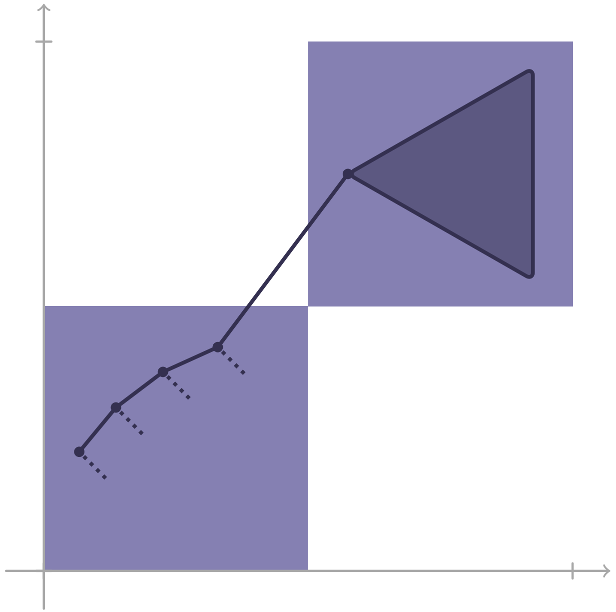

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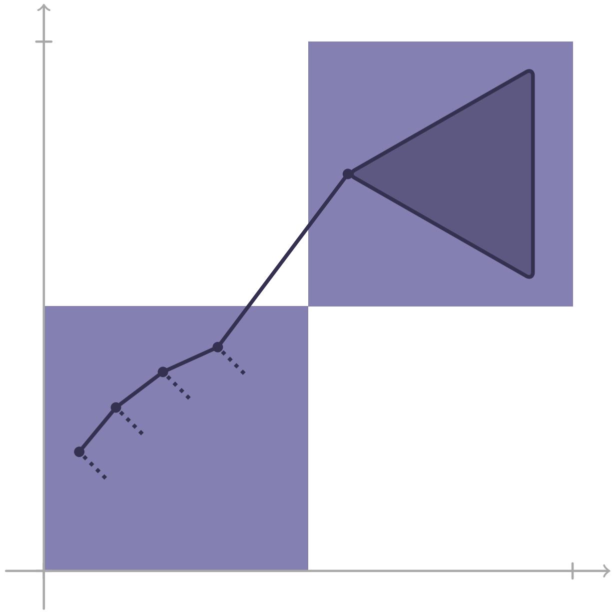

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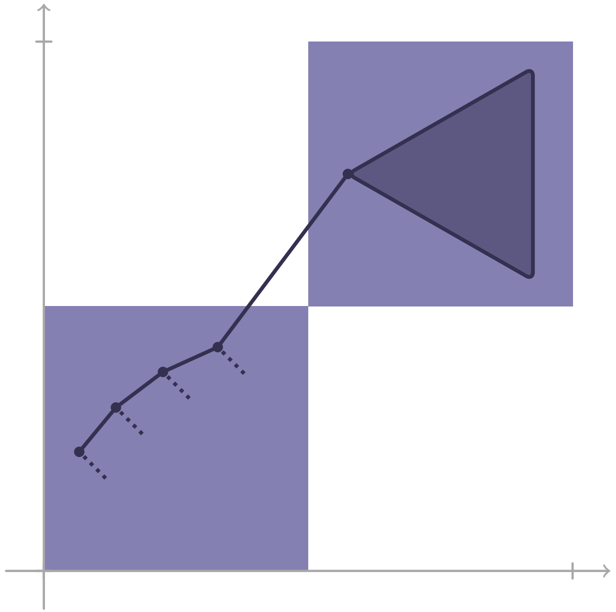

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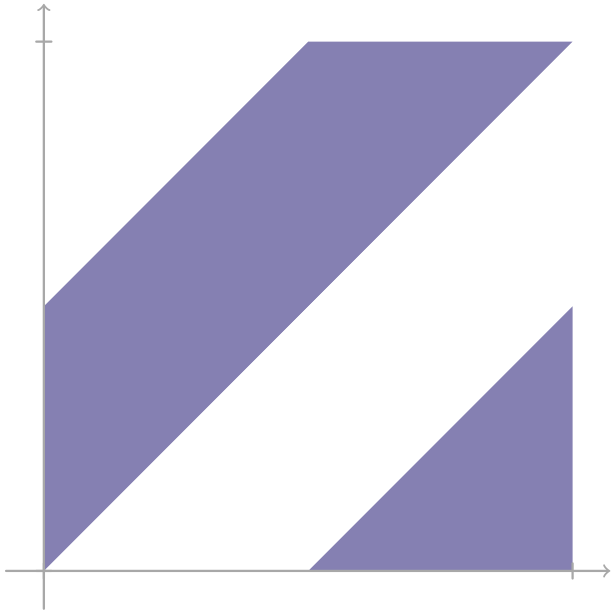

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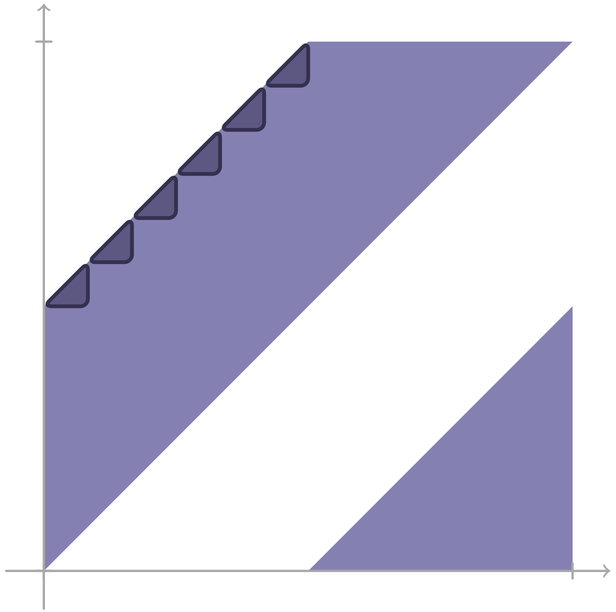

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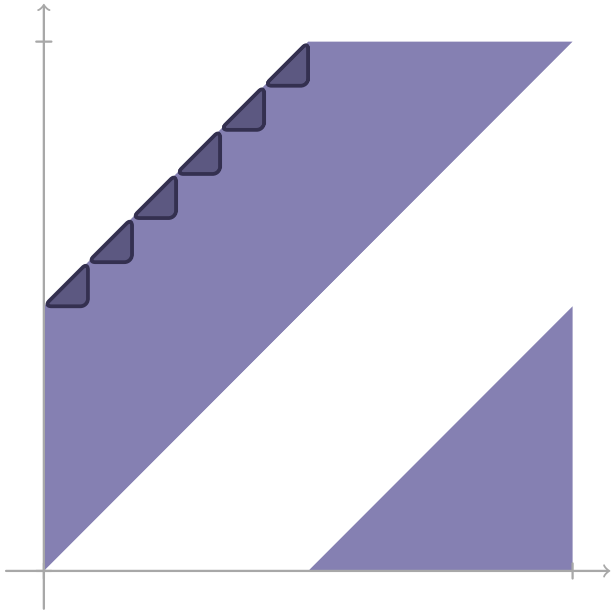

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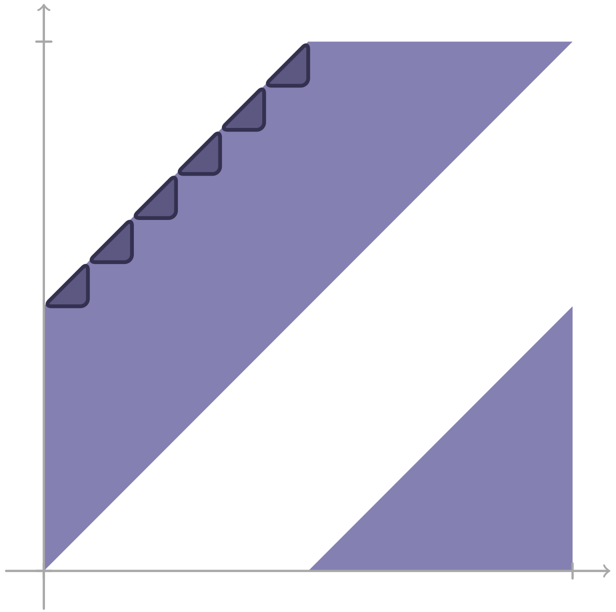

Condition: positive continuous left density

Consider the following distribution:

$$\rho(A) = 2\left[\left|A \cap \left\{(x,y) \in [0,1]^2 : y - x \in [0, 1/2] \bmod 1\right\}\right|_2\right].$$

In that case, we consider the top $\sqrt{n}/2$ triangles.

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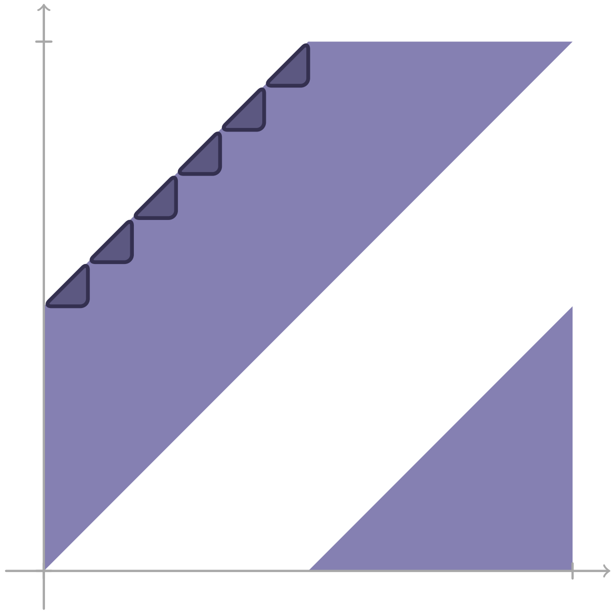
- The height is at least the number of triangles with 1 point.



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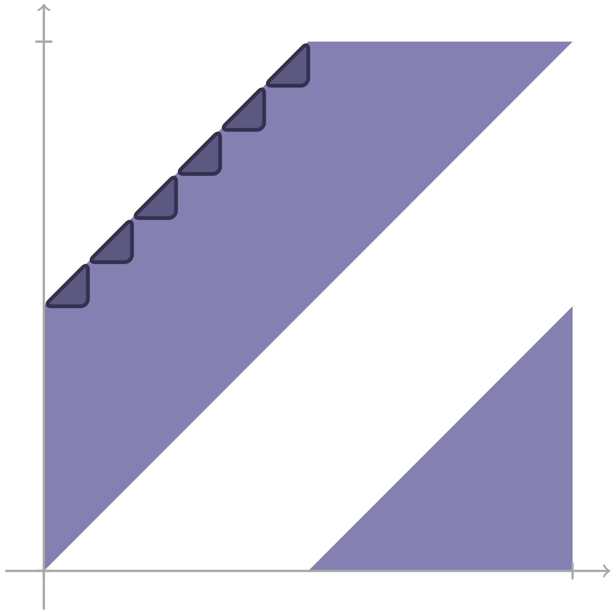

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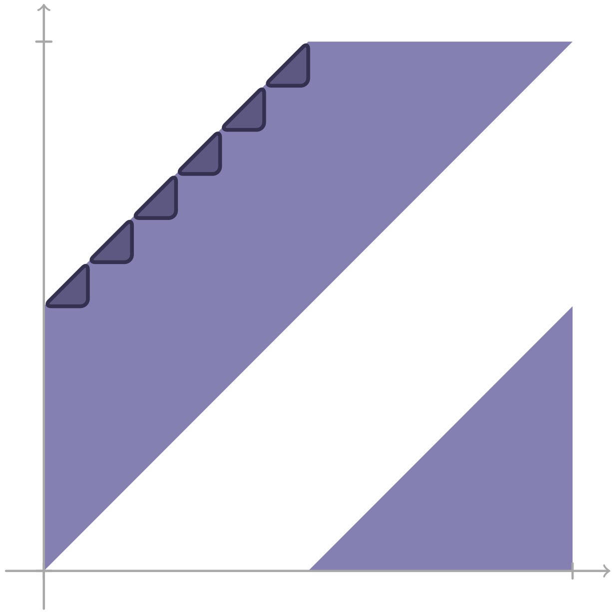

→ The height is at least of order $\sqrt{n}$.

→ By "smoothing" the density, we can get a height of order $n^{1/2-\delta}$.

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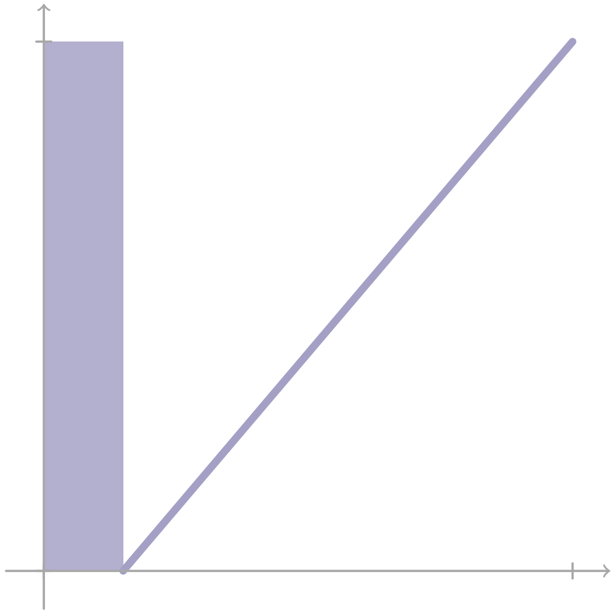

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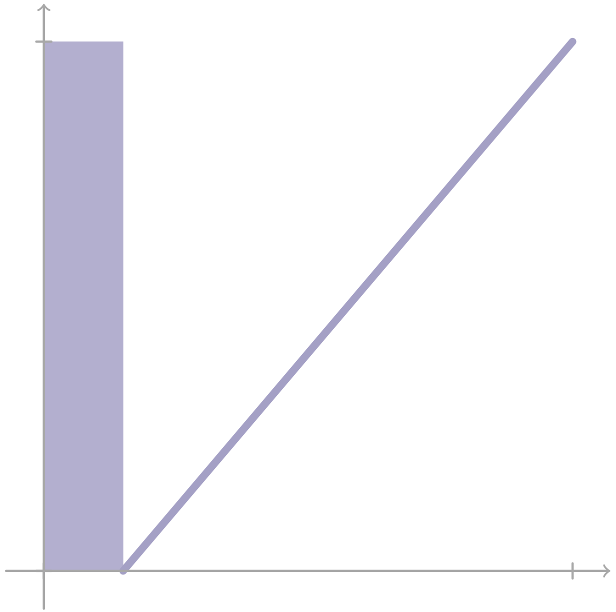

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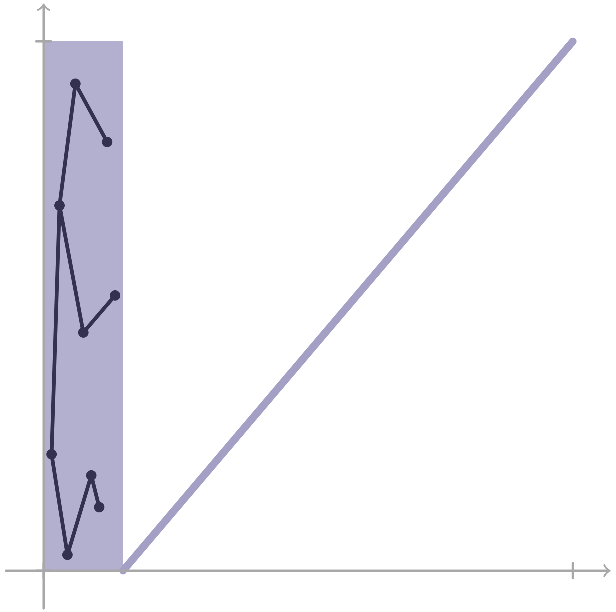

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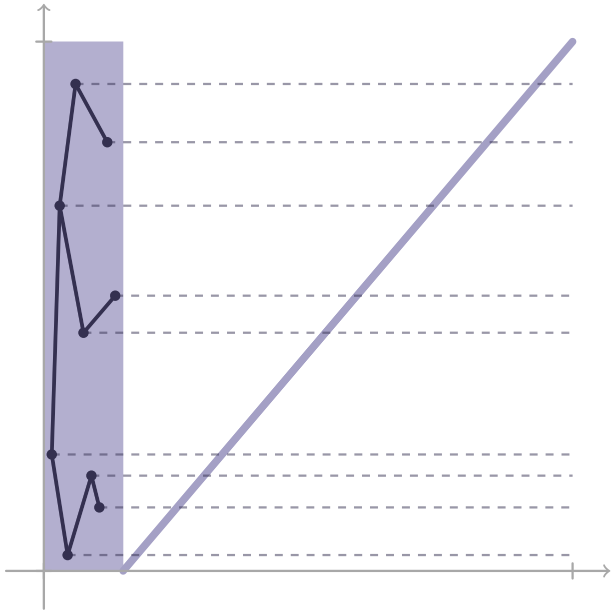

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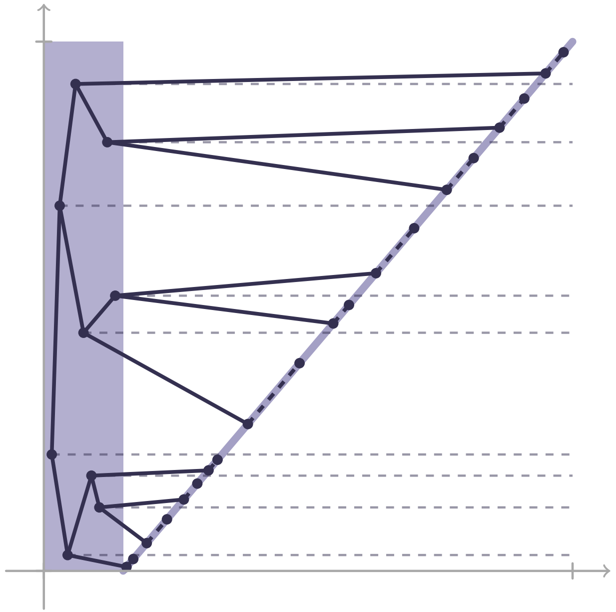

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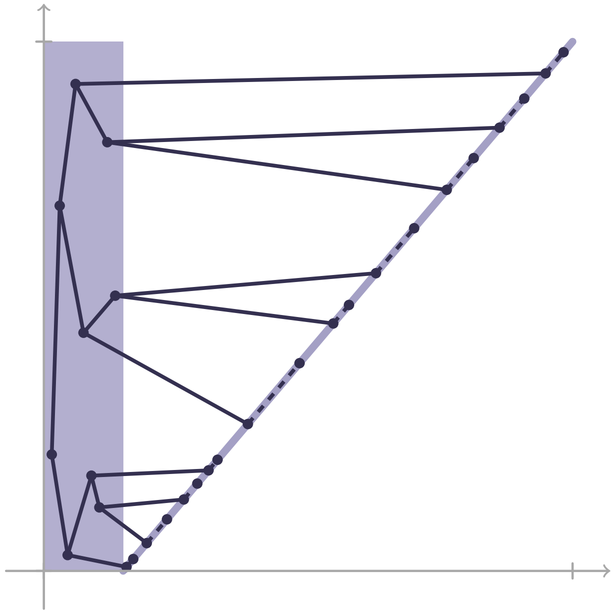

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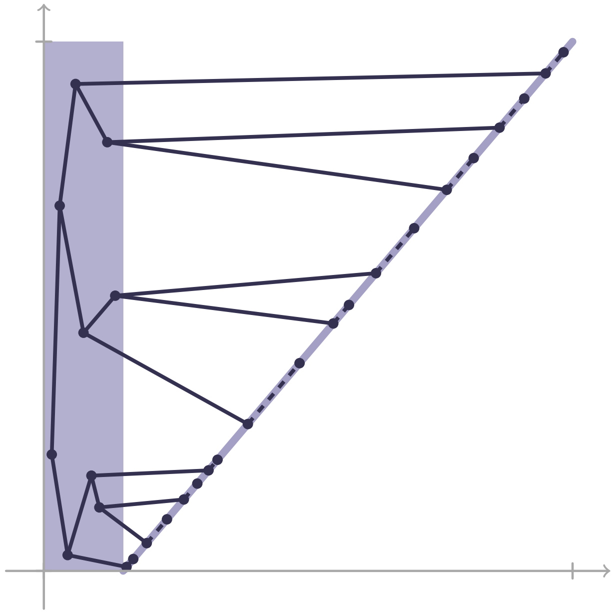

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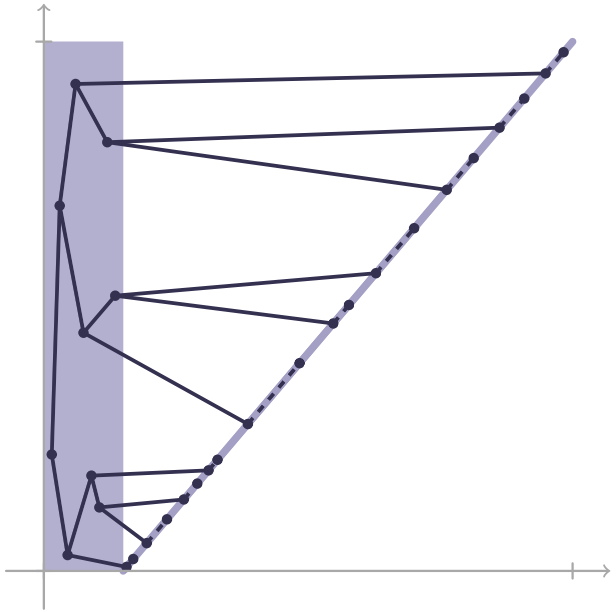

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$\rightarrow$ The height is at least $(1-\epsilon)\epsilon^{-1}\log n$.