Local weighted optimizations and open problems

Benoît Corsini
with L. Addario-Berry
and J. Barrett

Table of contents

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Motivation

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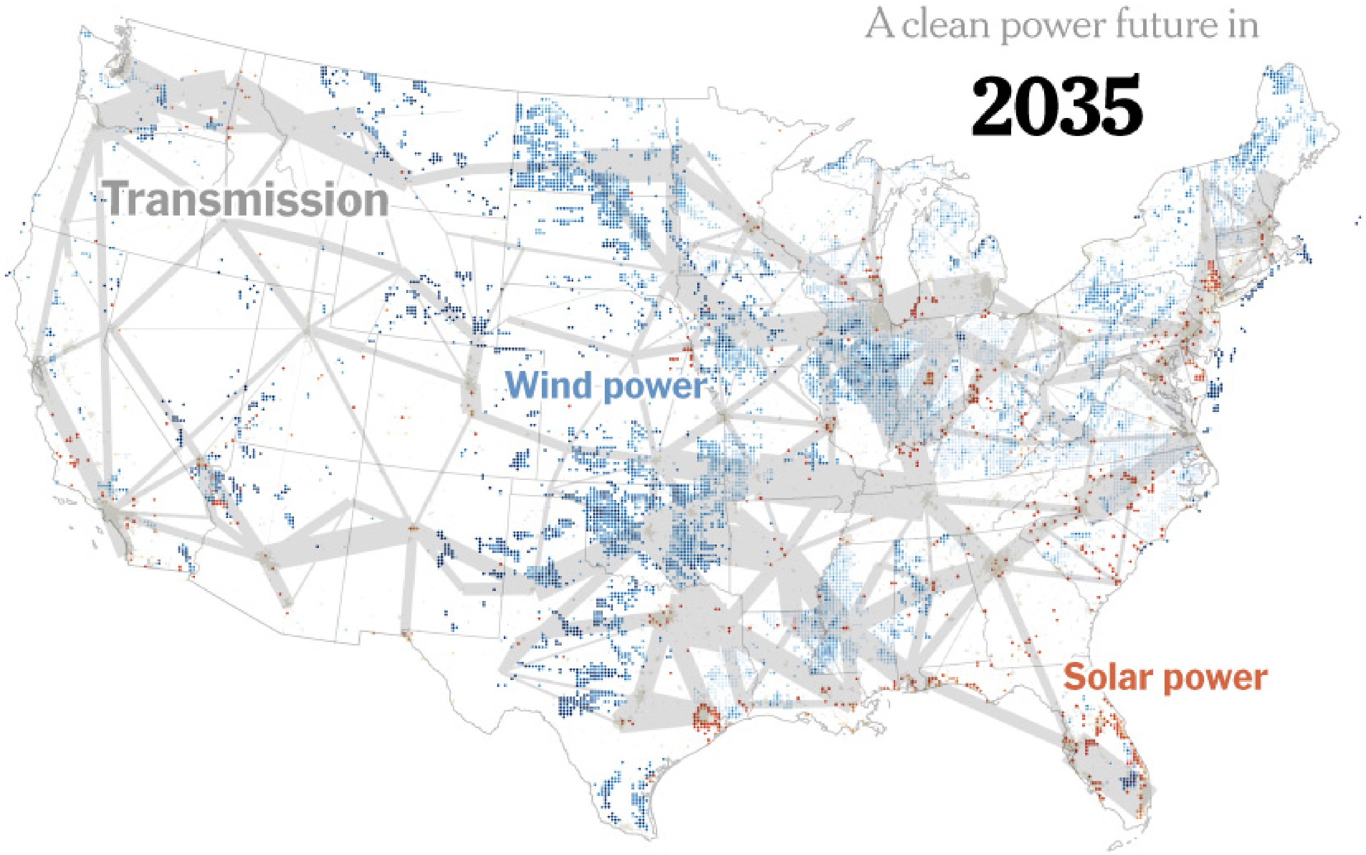
Assume we have a target property that we want our network to satisfy; can we operate "local" modifications eventually leading to the global graph satisfying this property?

Motivation



Source: mivolink.blogspot.com

Modernizing Canada's Aging Power Grid

by Powertec Electric | Apr 20, 2019 | Electrical Power, Electricians, Hiring Electricians | 0 comments

In the 70s and 80s, there was a lot of investment into electrical infrastructure in Canada. New technologies were demanding higher electrical capacity in homes, and the growth of Canada's large urban centres meant that demand was sure to remain high. The surge of investment into the grid was so monumental that supply actually ended up outweighing demand, and electricity could be bought on the cheap for many years. These investments have sustained us for quite some time, but we may now be reaching the breaking point of our electric grid.

Source: powertec.ca

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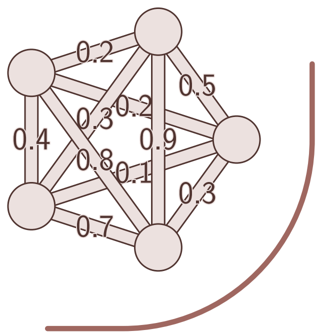

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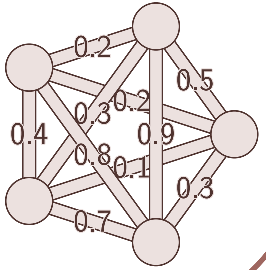
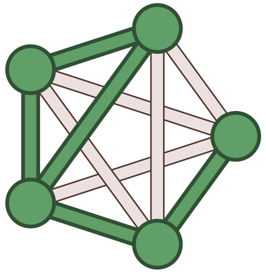


$H =$  $, \ \lambda = 1$

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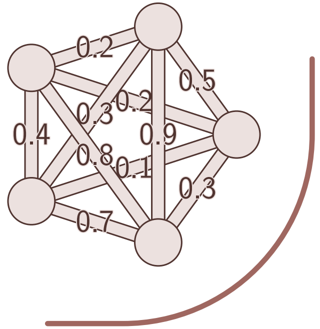
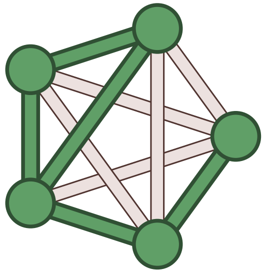
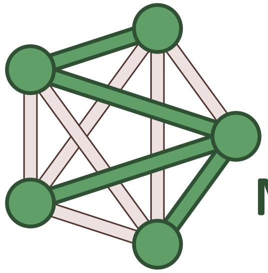


$H =$  $, \ \lambda = 1$



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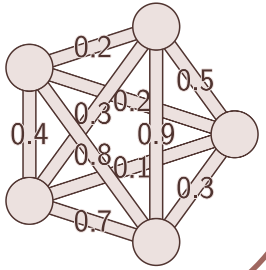
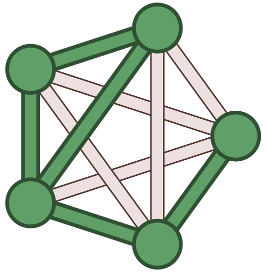
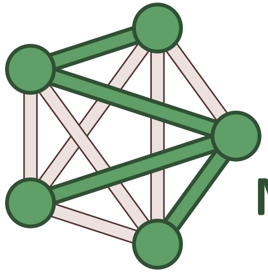


$H =$  $, \quad \lambda = 1 \quad \xrightarrow{\quad ? \quad}$  **MST**

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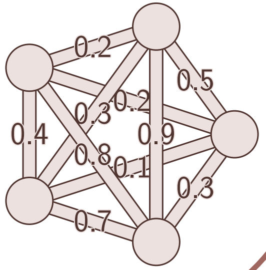
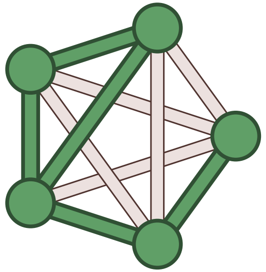
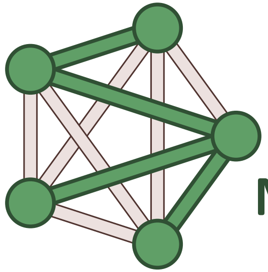
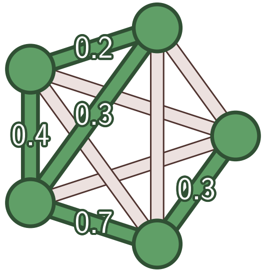


$H =$  $, \quad \lambda = 1$ $\xrightarrow{?}$  **MST**



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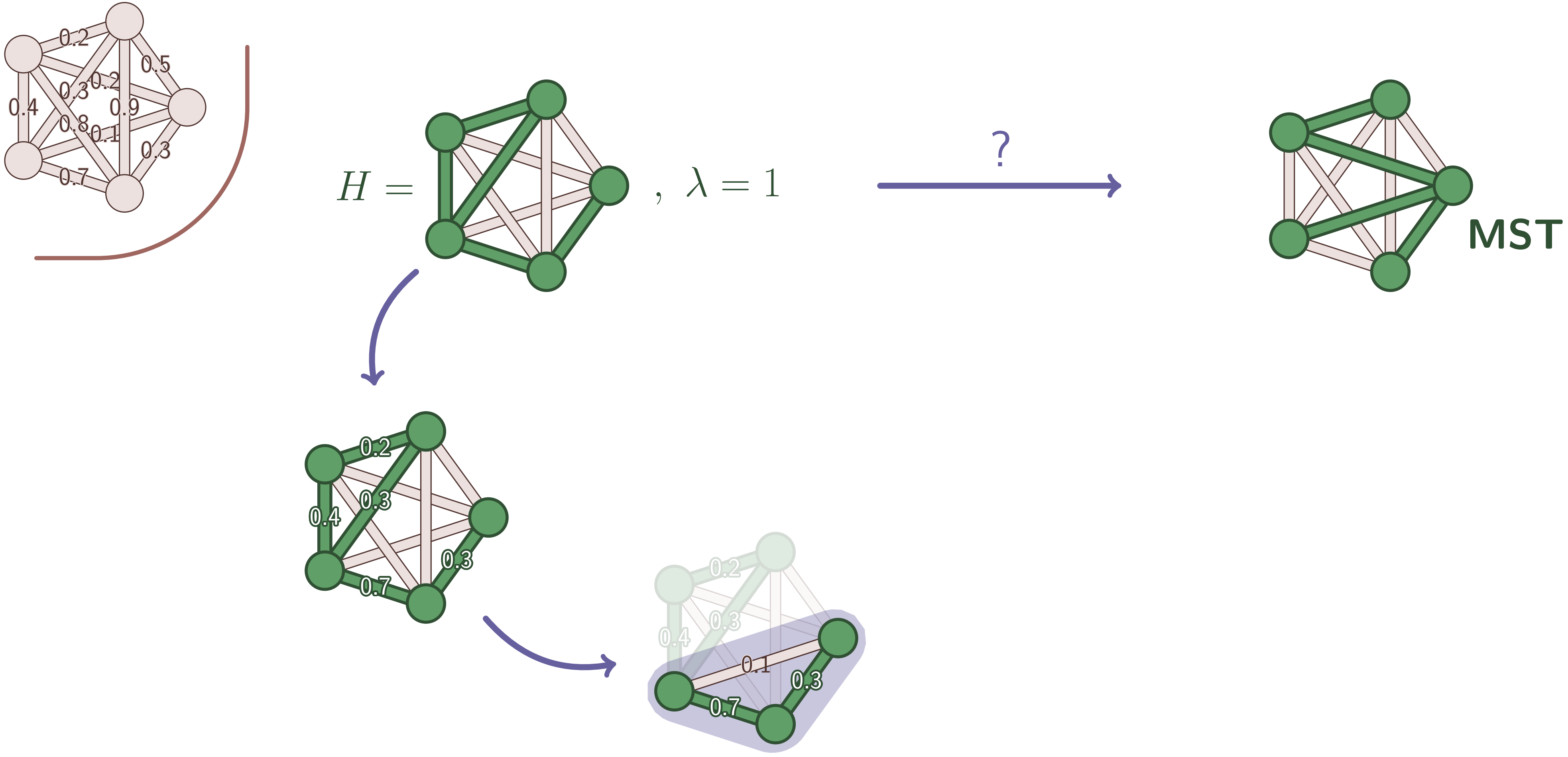
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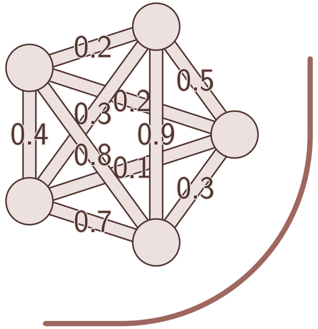
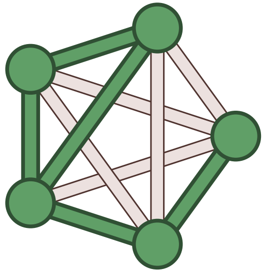


$H =$  $, \quad \lambda = 1$

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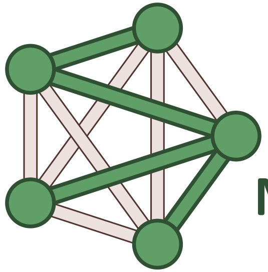

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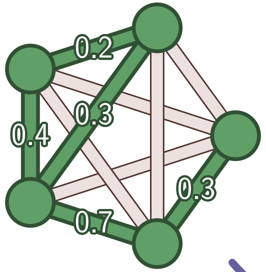

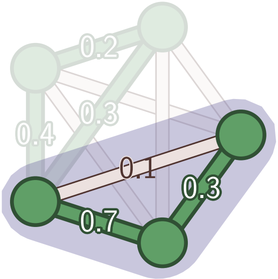

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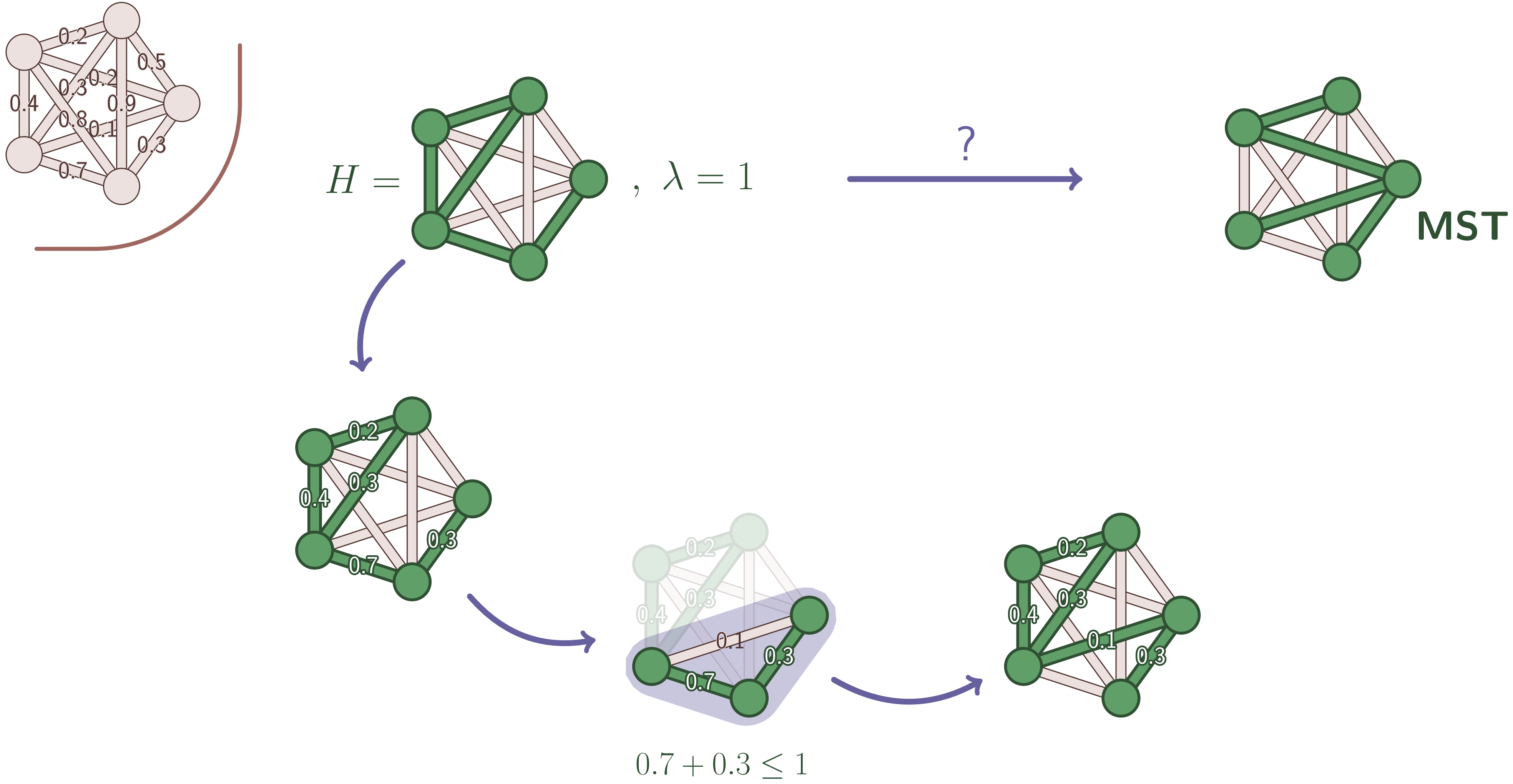

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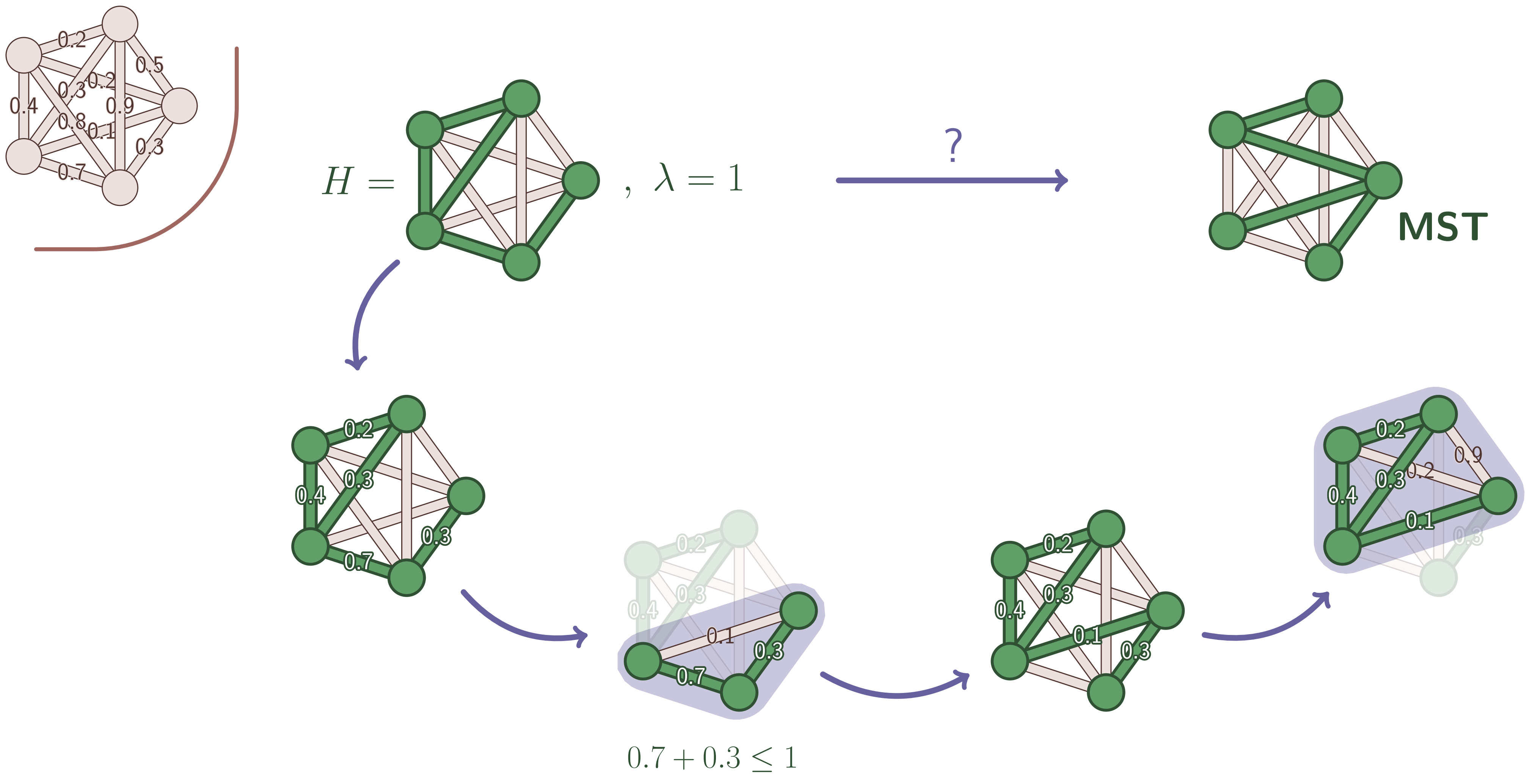

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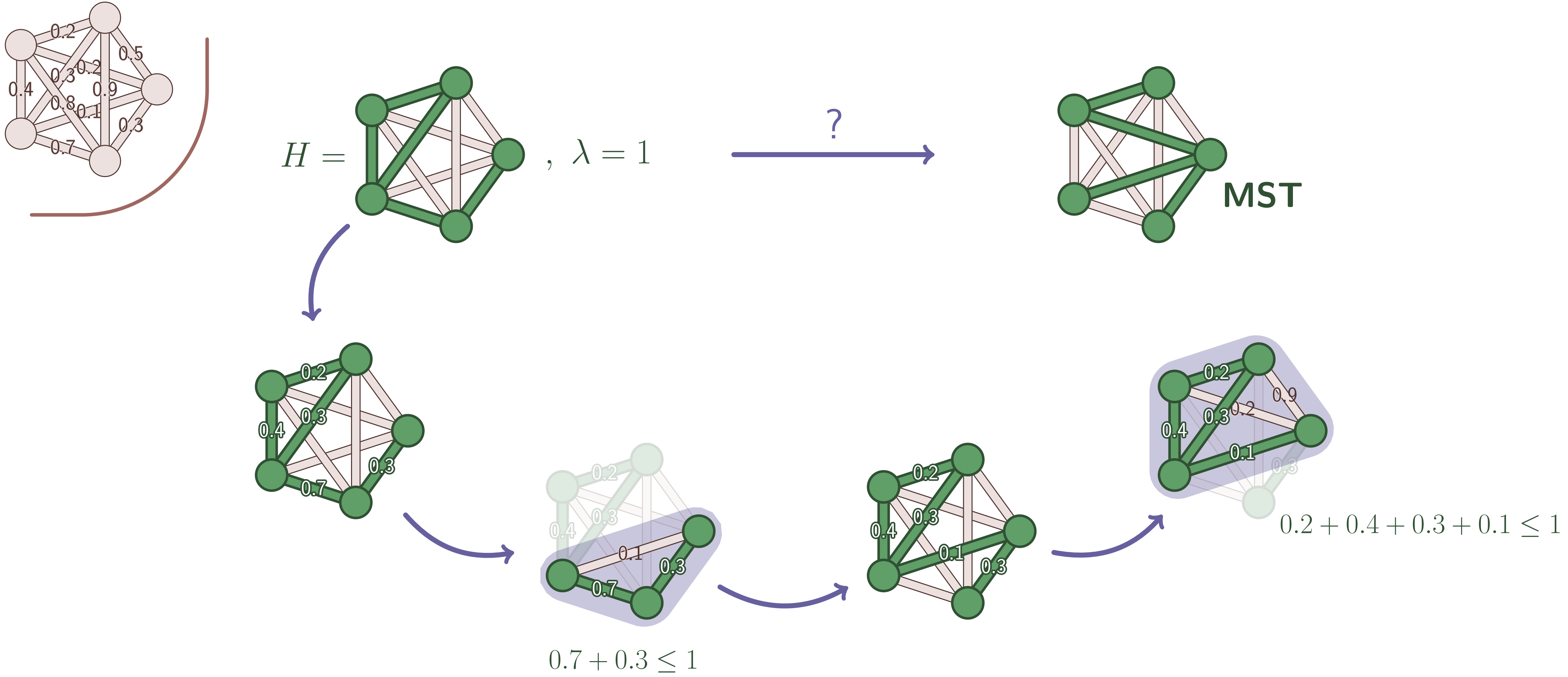

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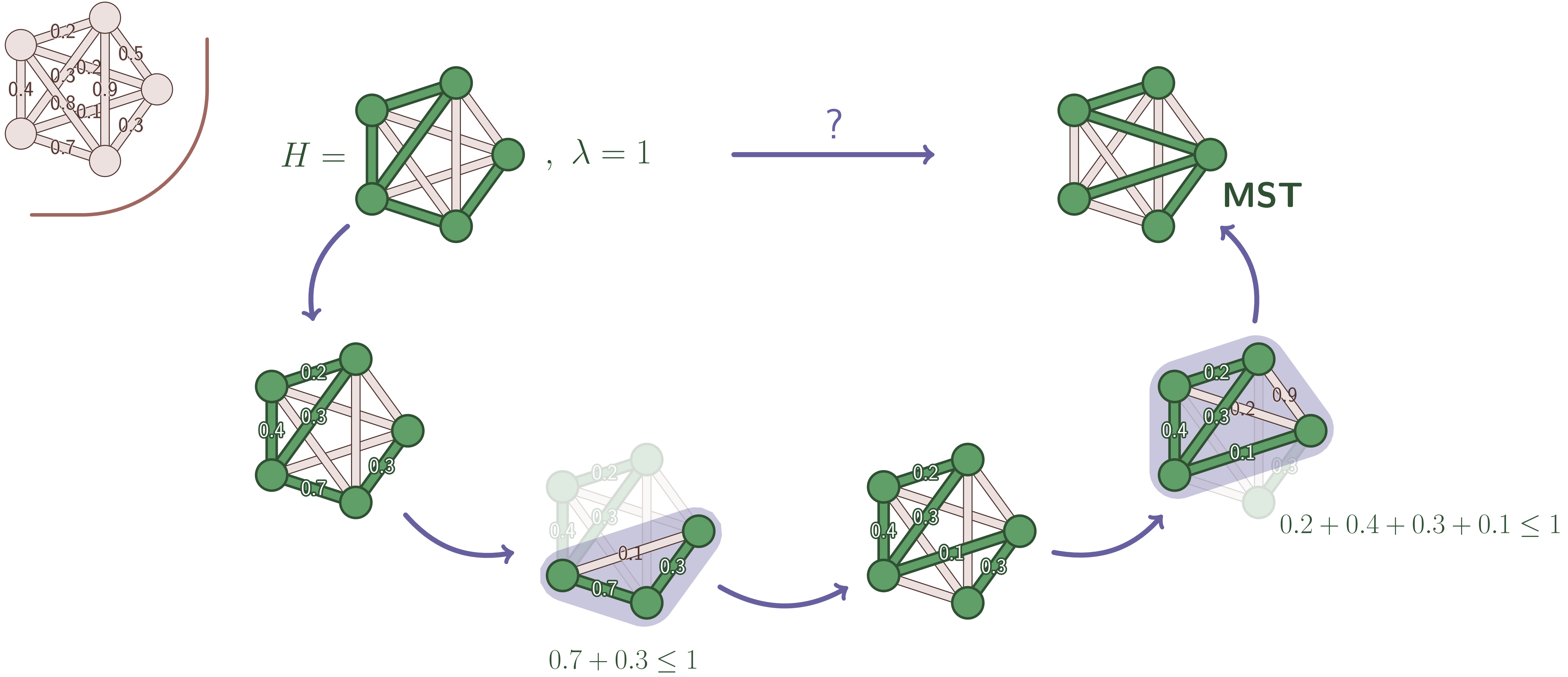
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→ We hope to characterize λ_{thr} when n is large for various choices of H_n.

Table of contents

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- there does not exist any optimization with respect to $(H_n, 1 - \varepsilon)$.

→ There is a universal threshold at 1, no matter the structure (or density) of H_n.

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→ There does not exist an optimization with respect to $(H_n, 1 - \epsilon)$.

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→ Can we show that the maximal weight of these sets is not too large (i.e. $\leq 1 + \epsilon$)?

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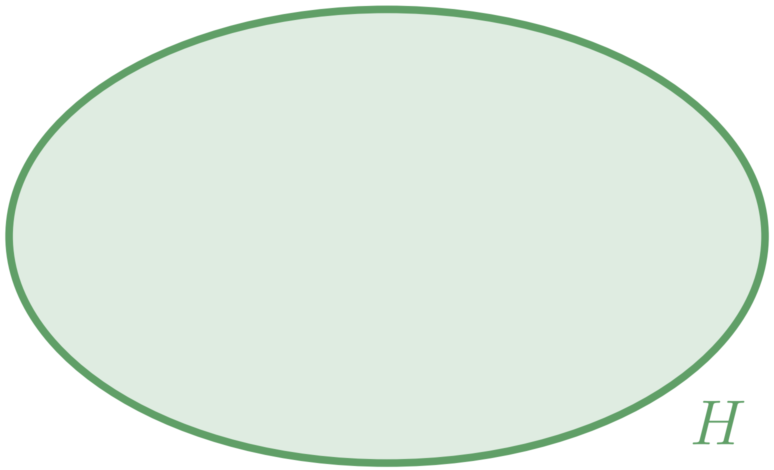

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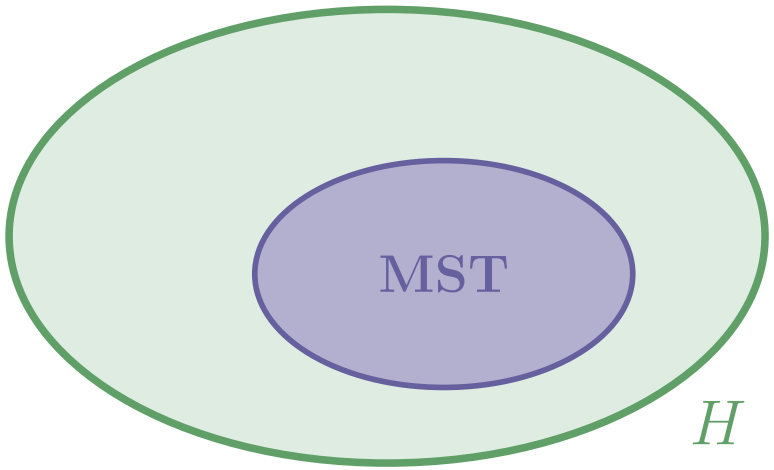

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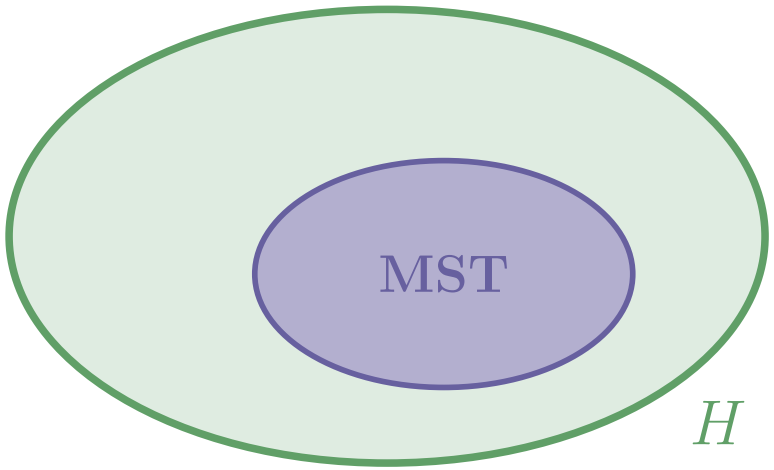

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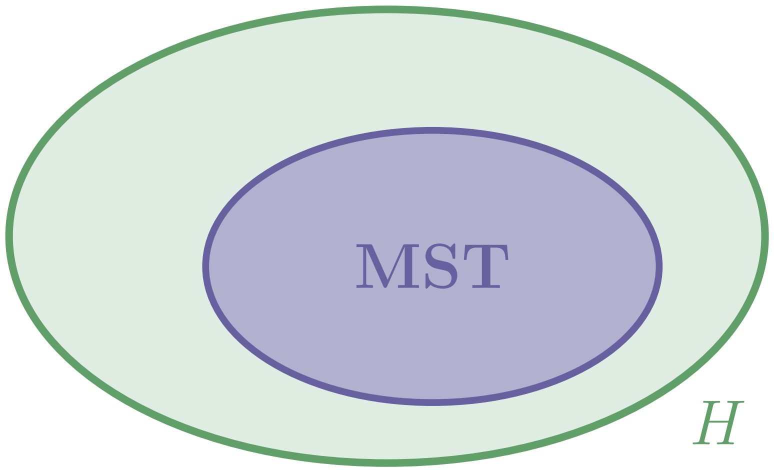

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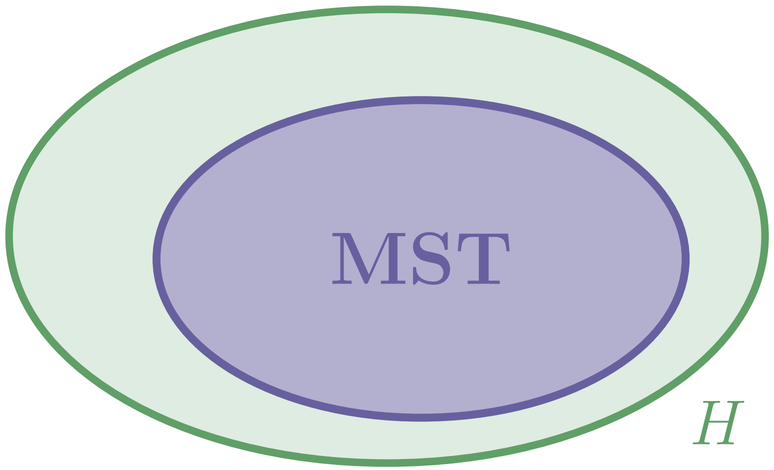

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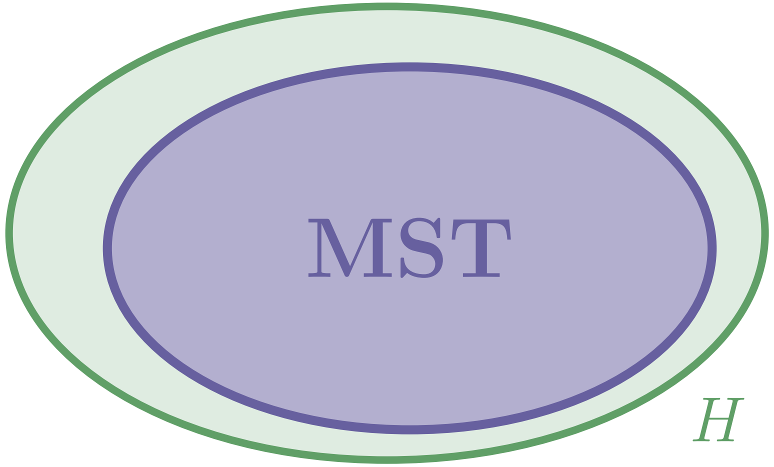

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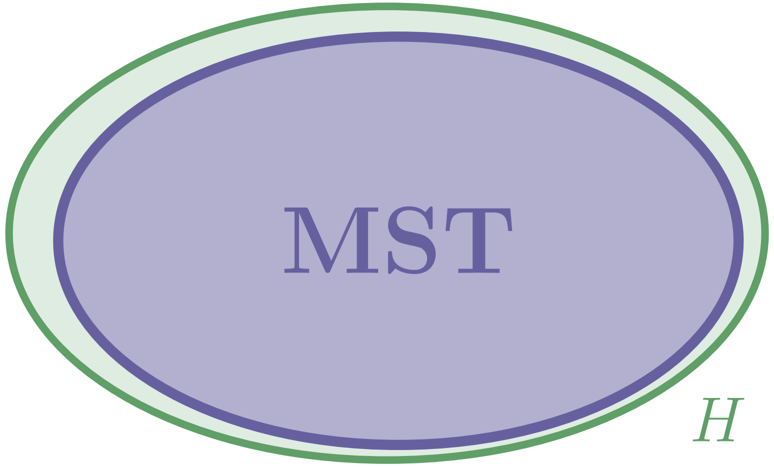

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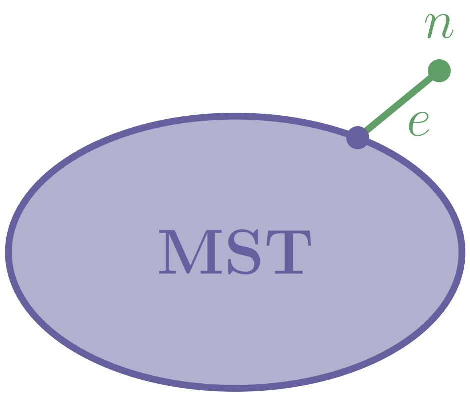

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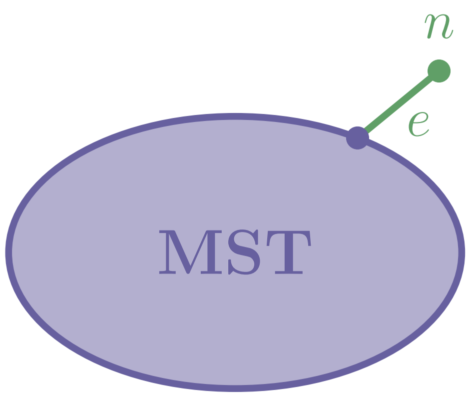

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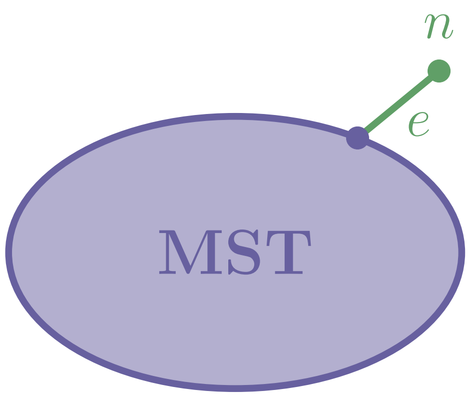

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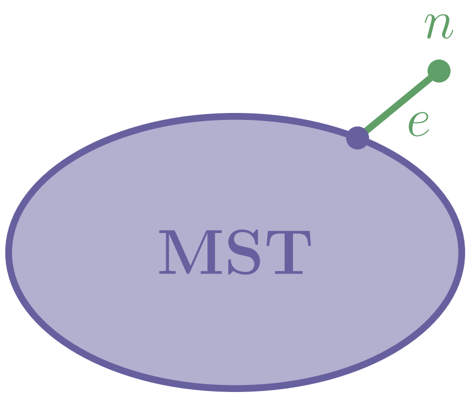

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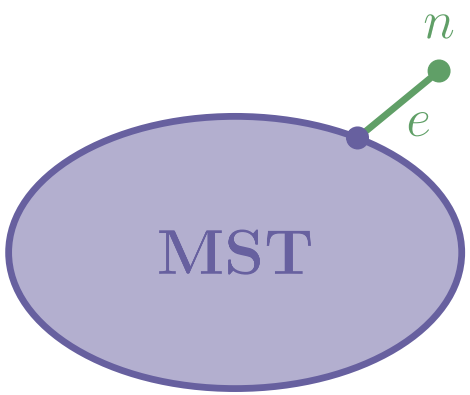

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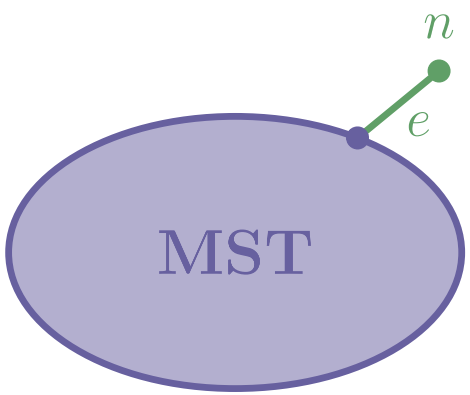

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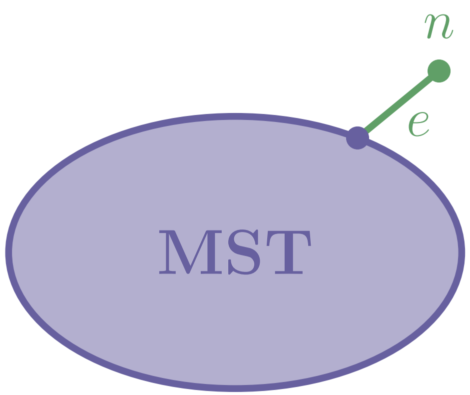

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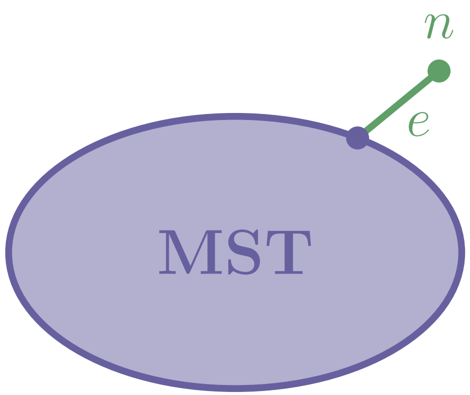

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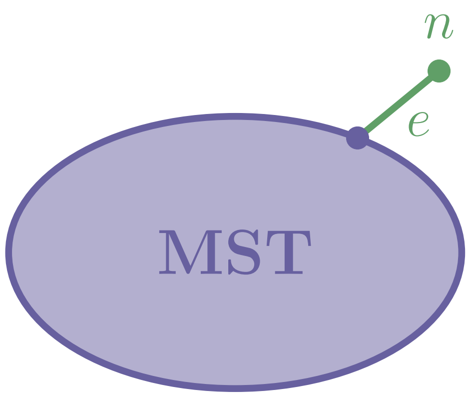

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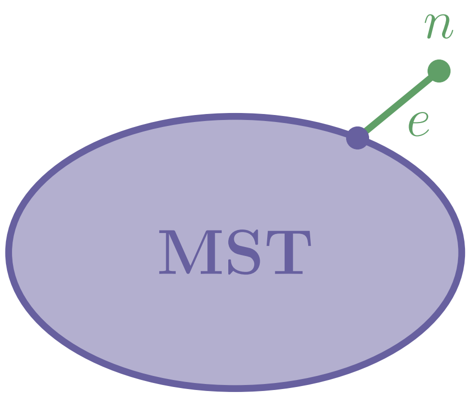

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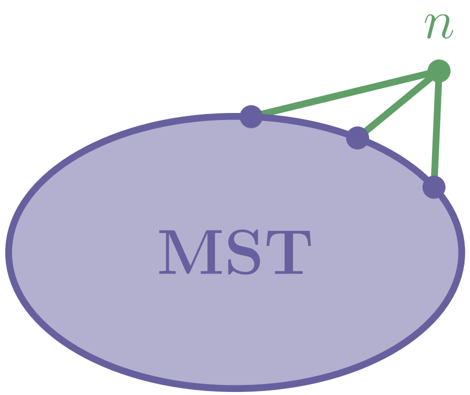

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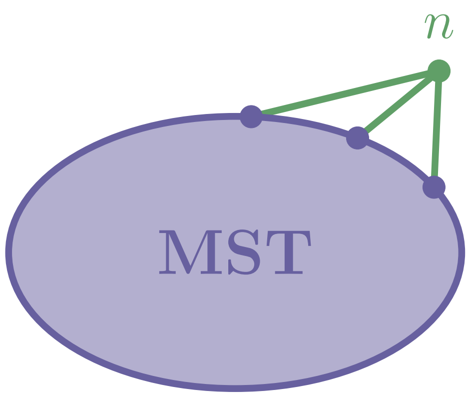

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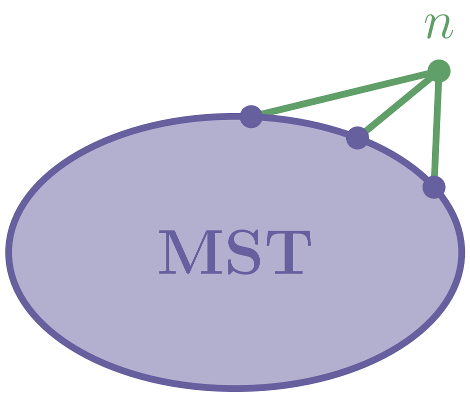

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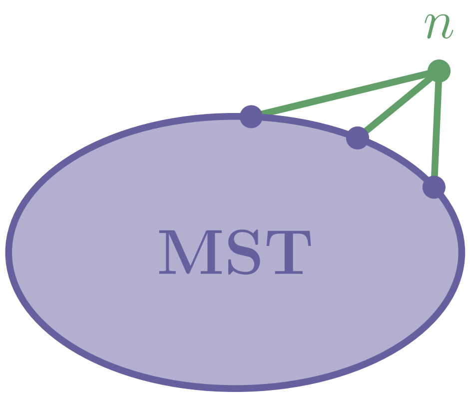

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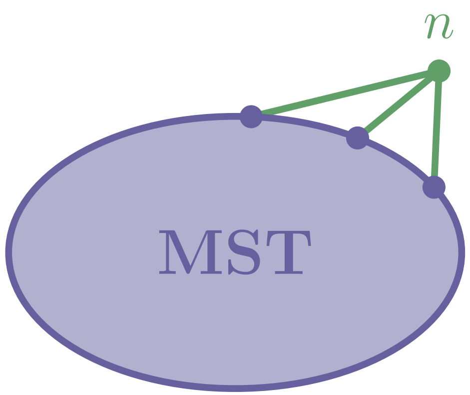

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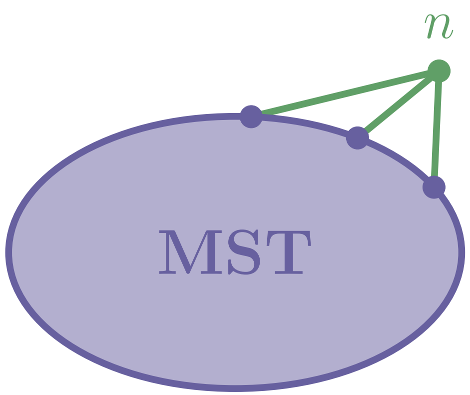

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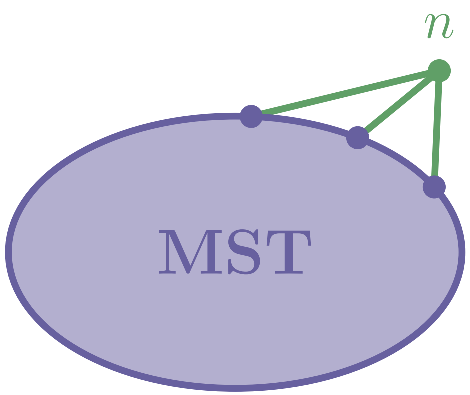

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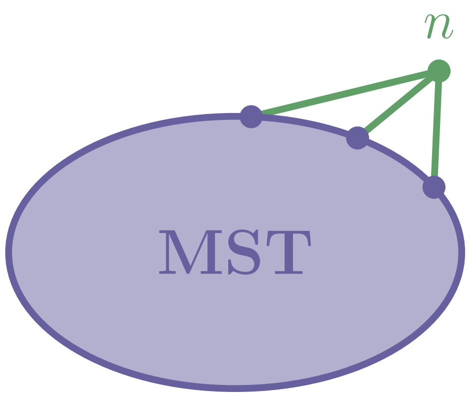

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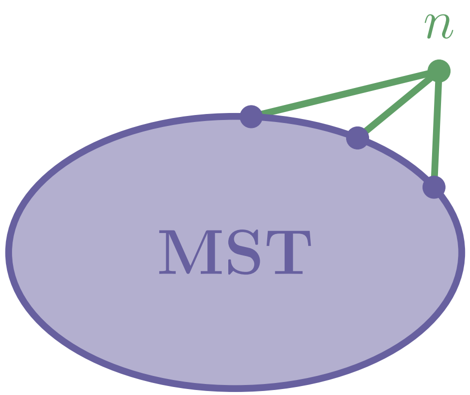
→ Luckily, we obtain steps with weight $1 + o_{\mathbb{P}}(1)$ again.



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⚠ We need to be careful on the dependency with the edge weights $\mathbb{U}$.

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→ In that case, the eating algorithm still works, but the proof is more tedious.

Table of contents

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$\rightarrow$ Let me focus on the first question, in particular the reason why it is and not .

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$\rightarrow$ We are now interested in the behaviour of the size when we put more constraints on the partition.

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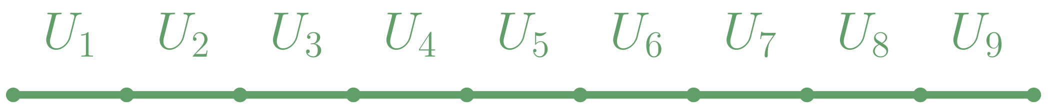

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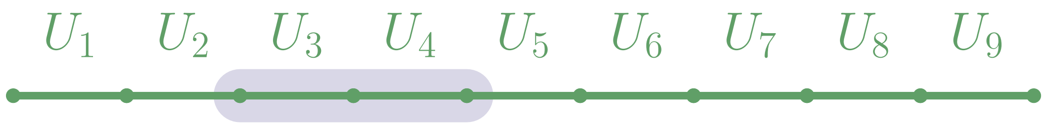

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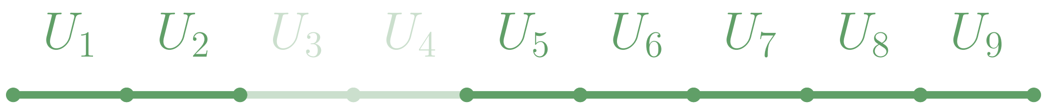

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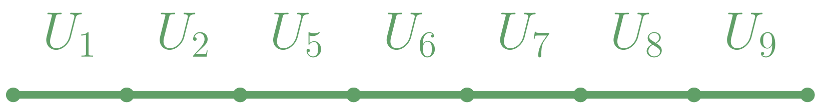

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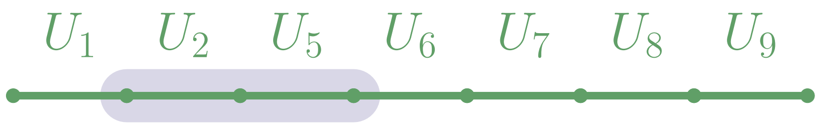

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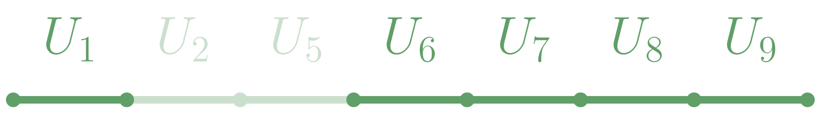

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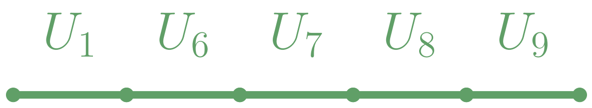

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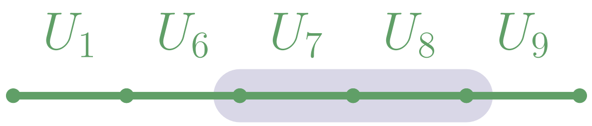

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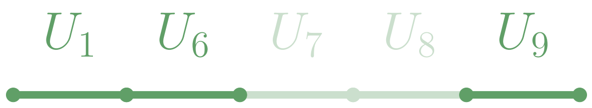

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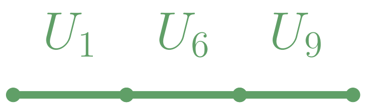

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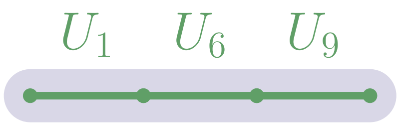

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In particular, is it approximately $n/2$ as it was the case for general partitions?

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It is actually rather easy to prove the lower bound. Indeed, for a partition $(S_1, \ldots, S_k)$ of weight at most 1, we have

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$\rightarrow$ I personally tend to believe that the correct behaviour is $n/2$ for line-connected partitions.

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Sadly, the upper bound proves to be more difficult to obtain. For example, if considering the special case of interval partitions (which are also themselves line-connected partitions, but easier to study), computations seems to show that the asymptotic size is of order $n/(2-\alpha)$ for some $\alpha > 0$.

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→ It would however lead to further questions...

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To fully solve the "speed problem", we would further need to understand the size of a line-connected partition of weight at most λ, which should not be substantially harder than the case $\lambda = 1$.

Thank you!

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References

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