Binary Search Trees

Benoît Corsini

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Where are you from?

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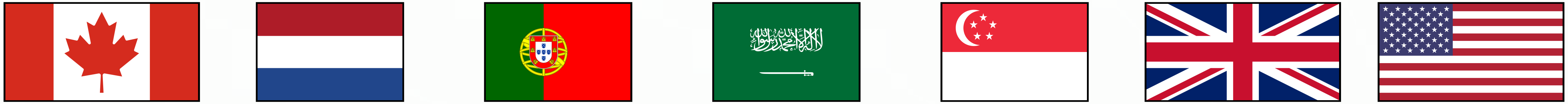

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Q: How long to go through the list in order?

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→ On average 4 queries to find the answer.

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  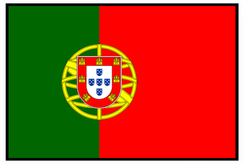 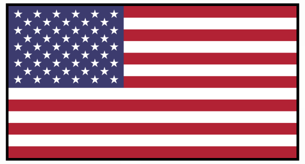 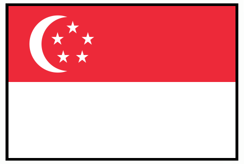  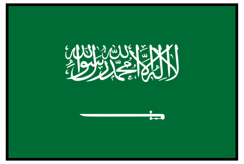

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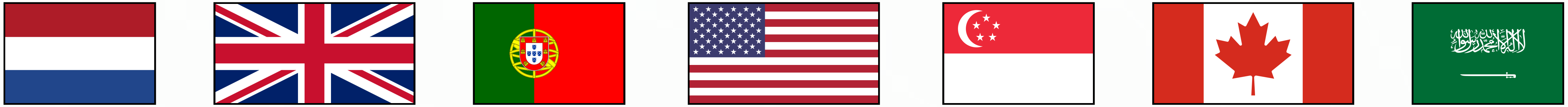

18M 68M 11M 340M 6M 42M 32M

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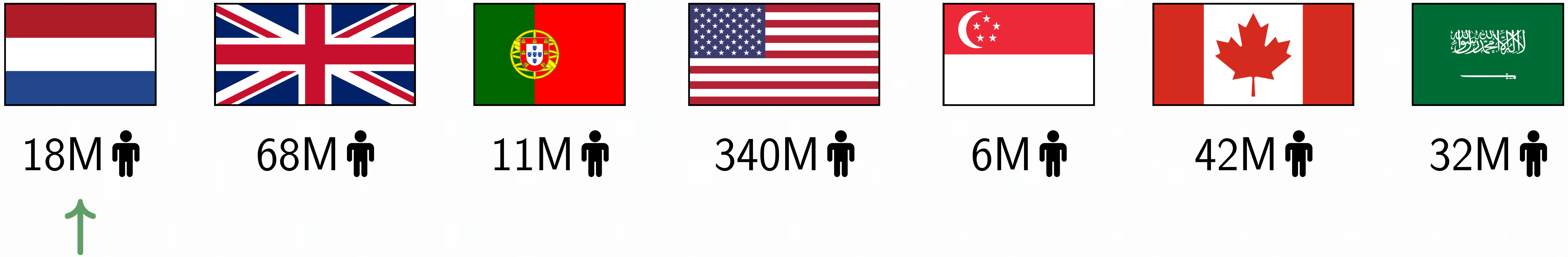

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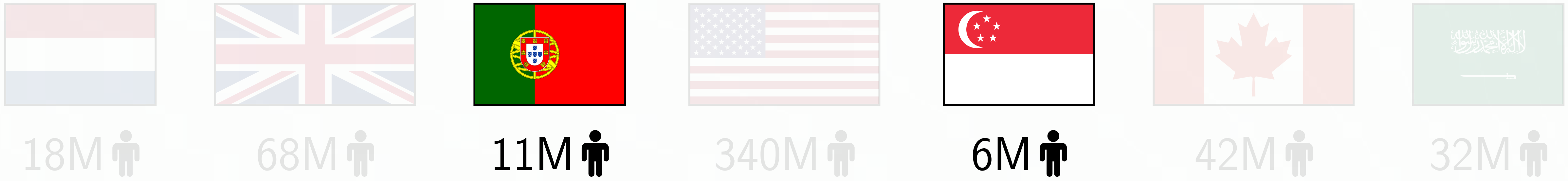

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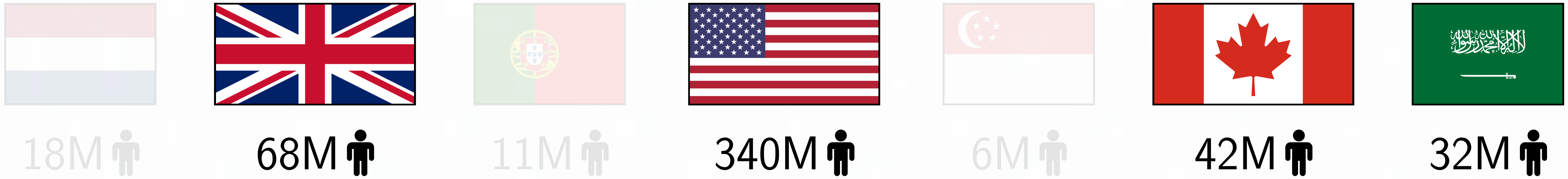

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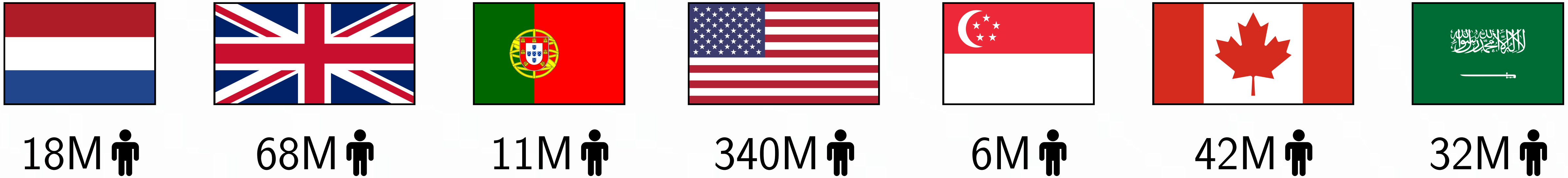

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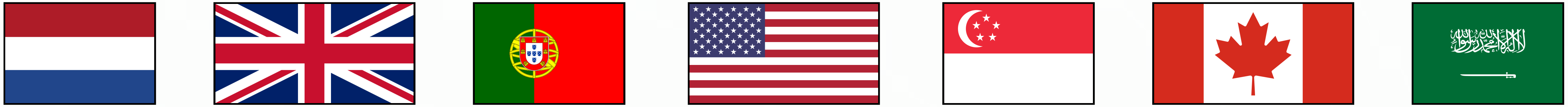

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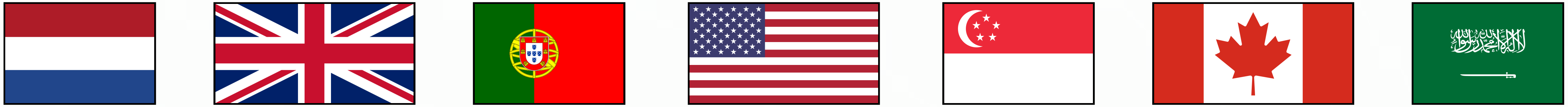

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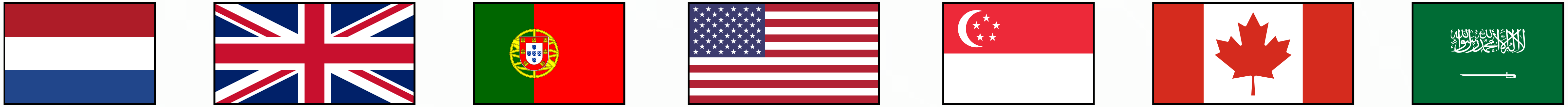

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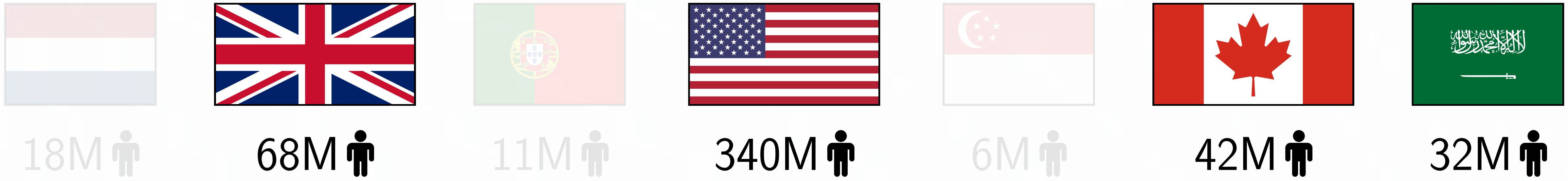

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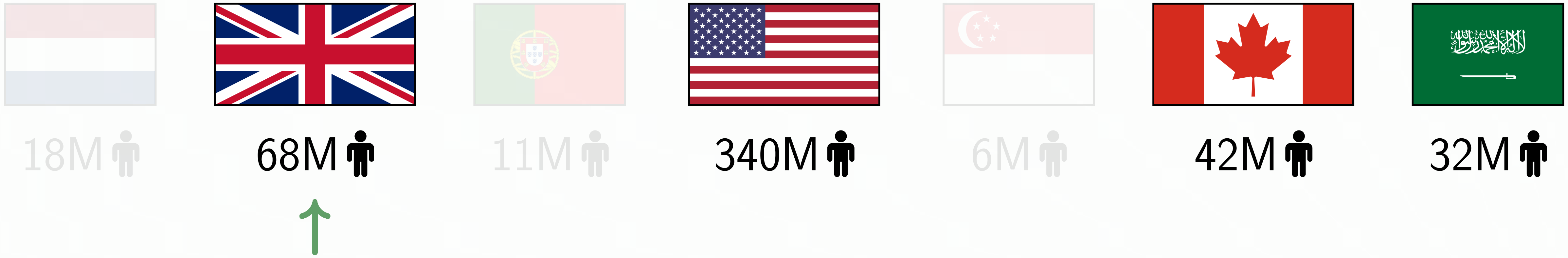

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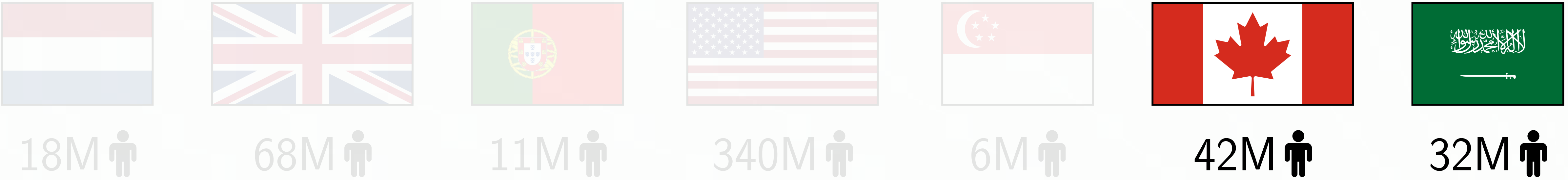

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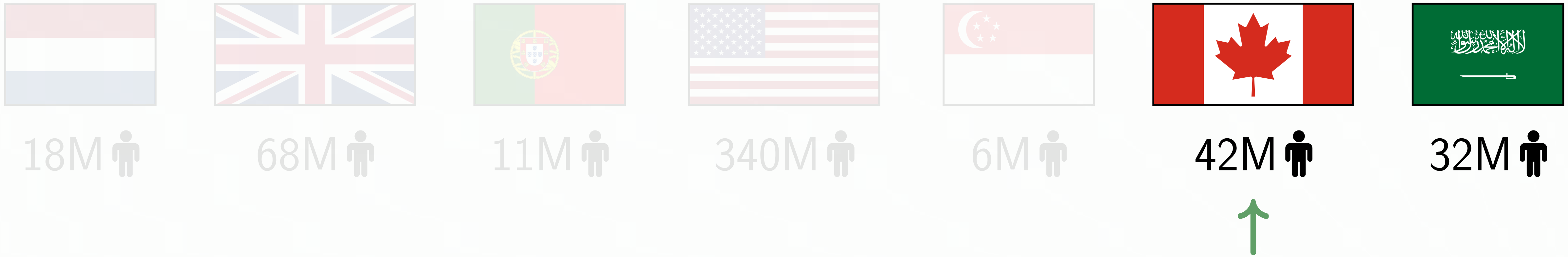

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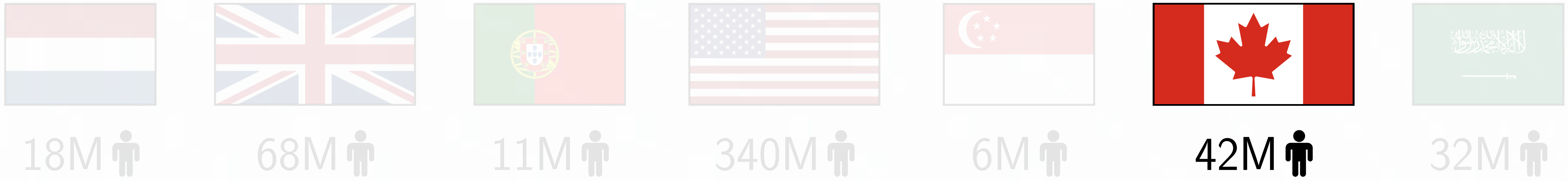

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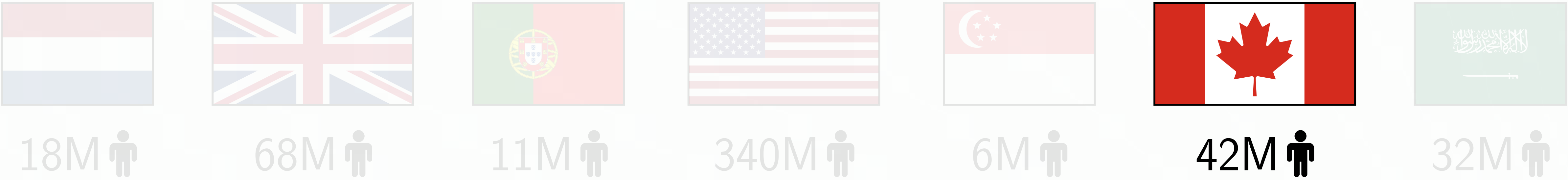

→ In this case, it took 3 queries using the population, against 6 not using it.

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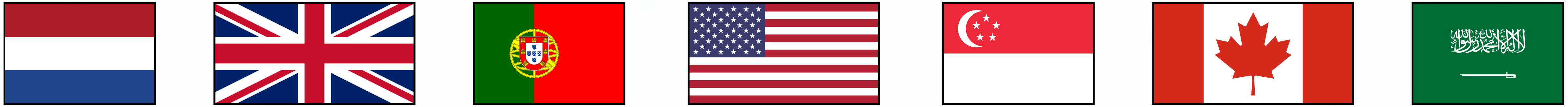

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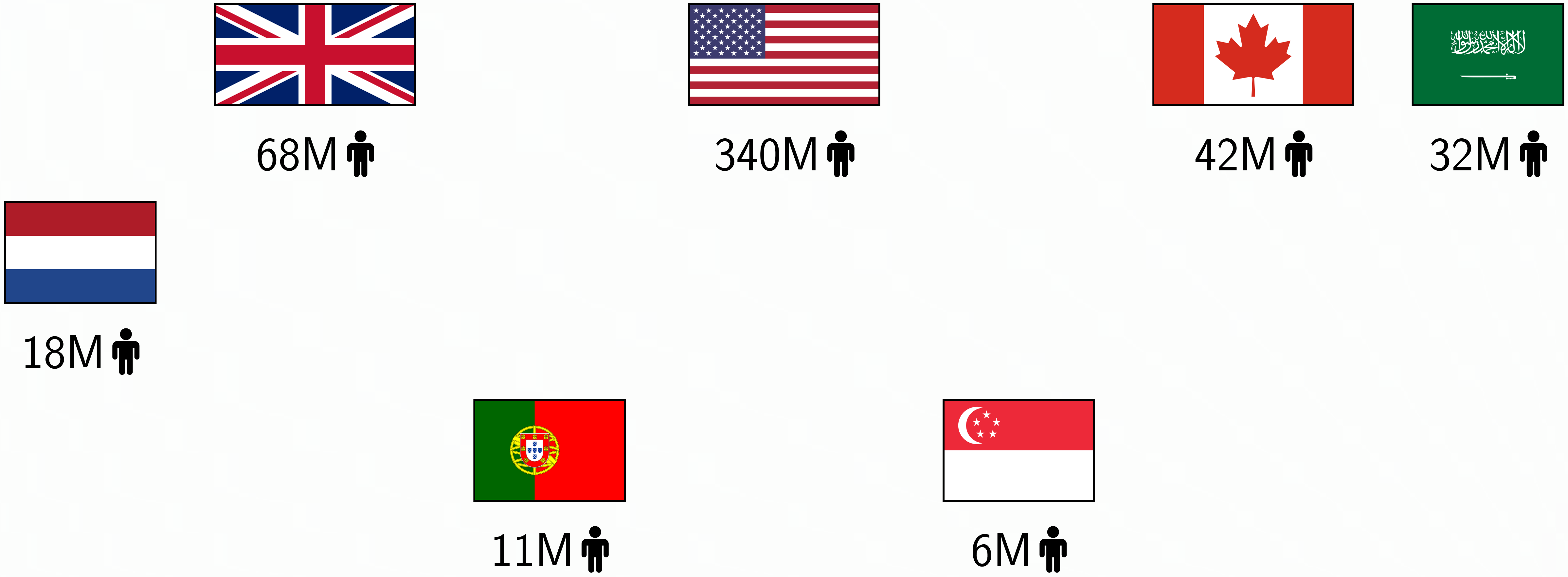

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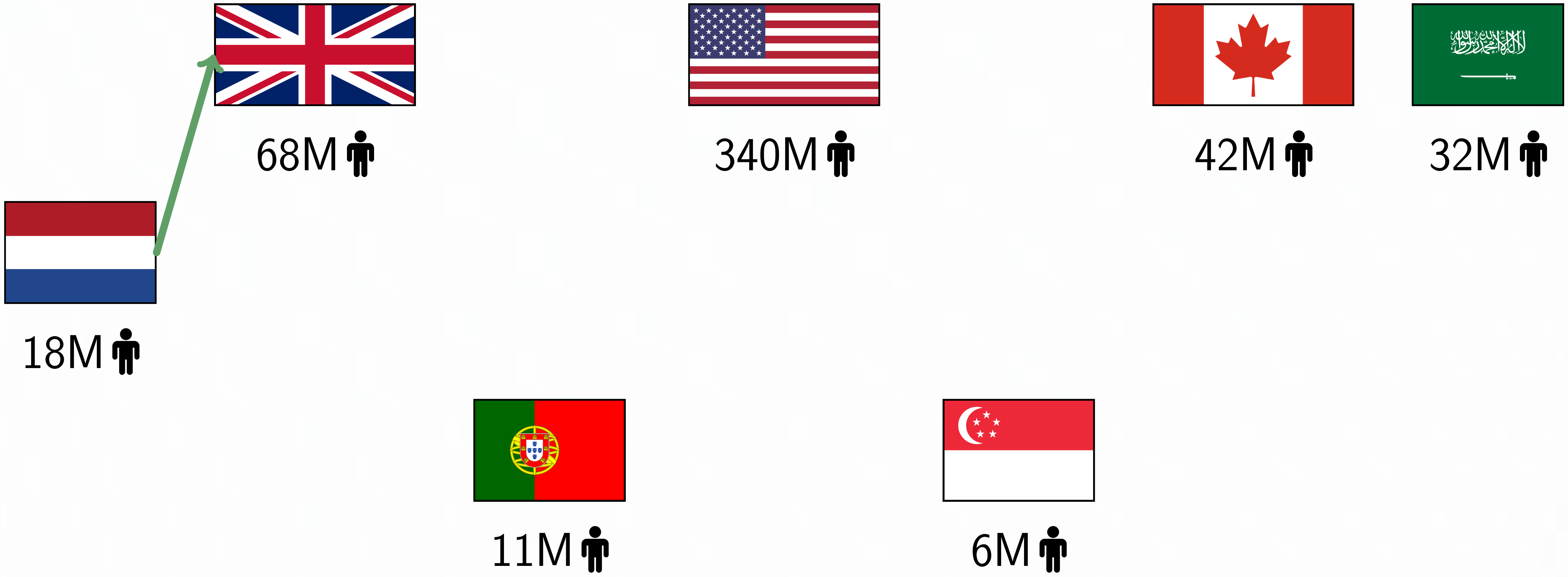

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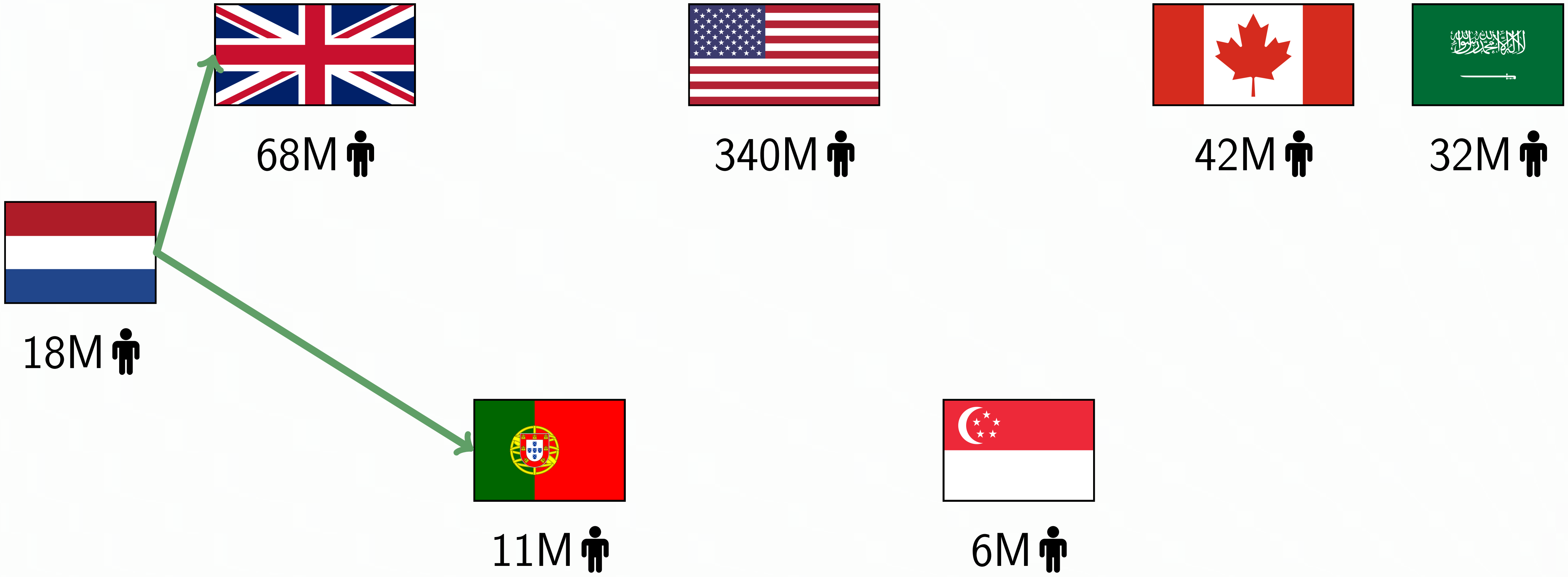

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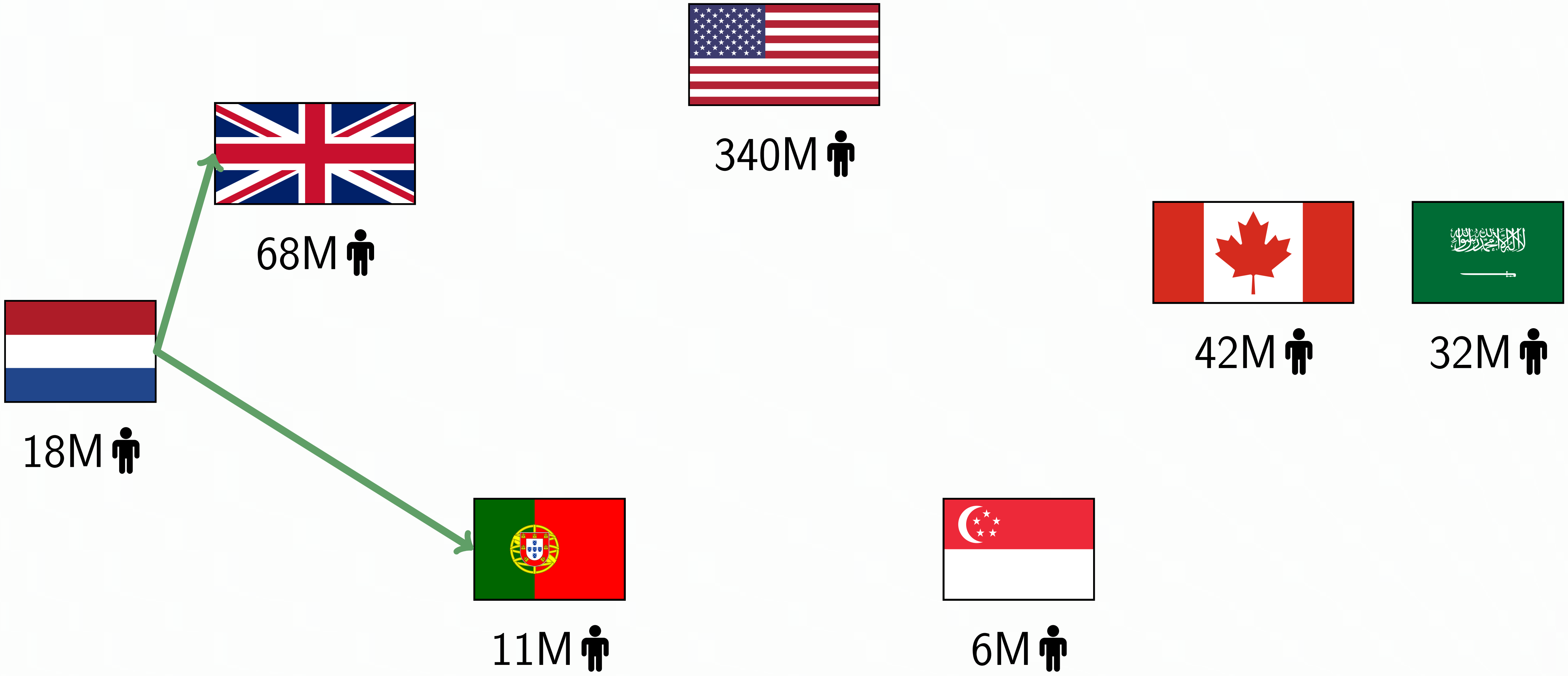

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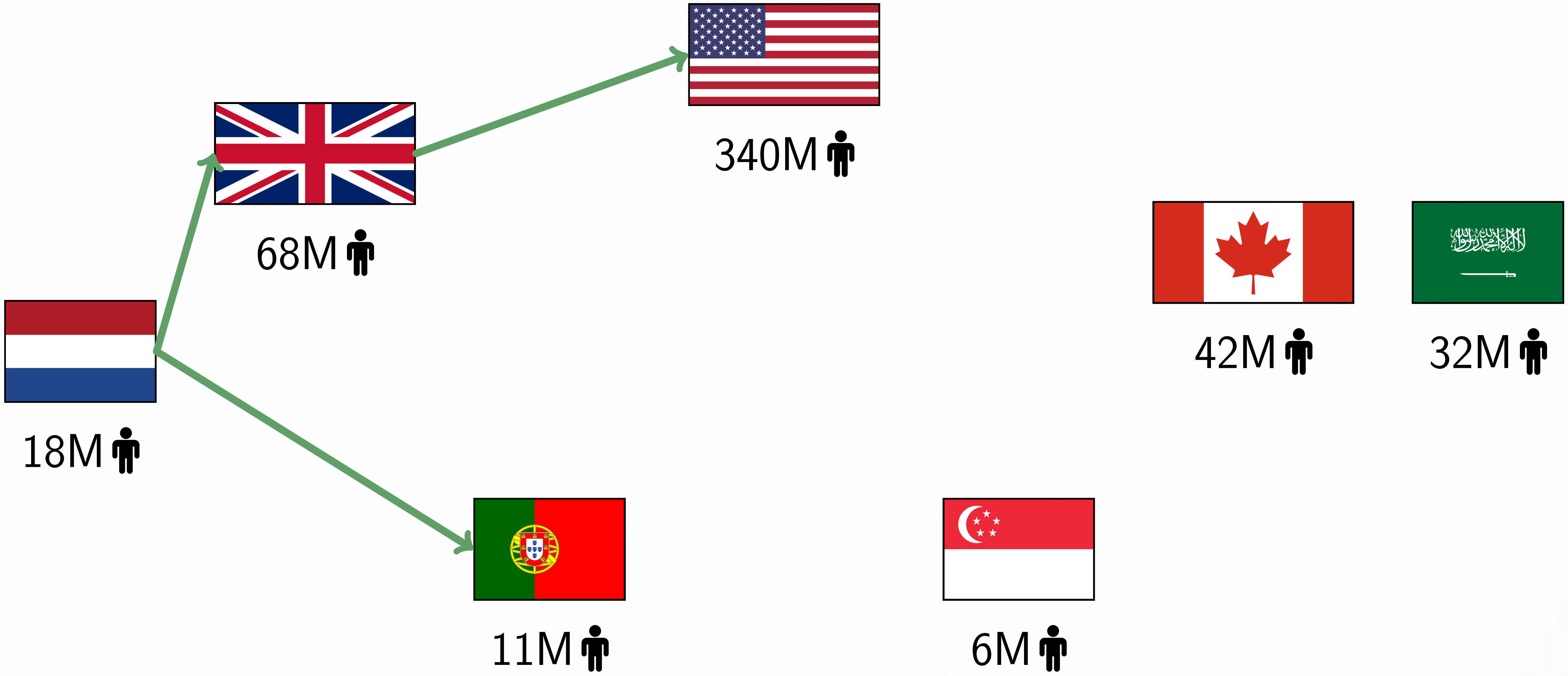

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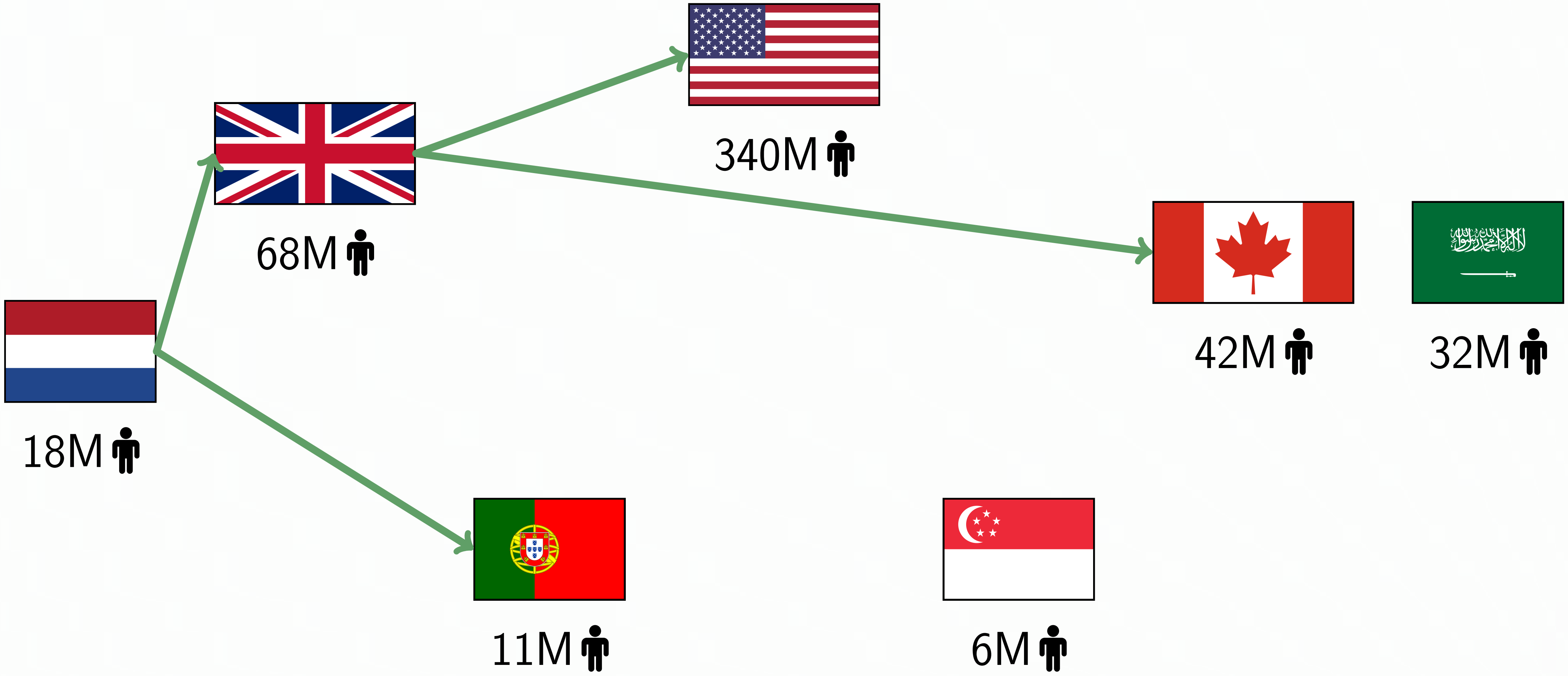

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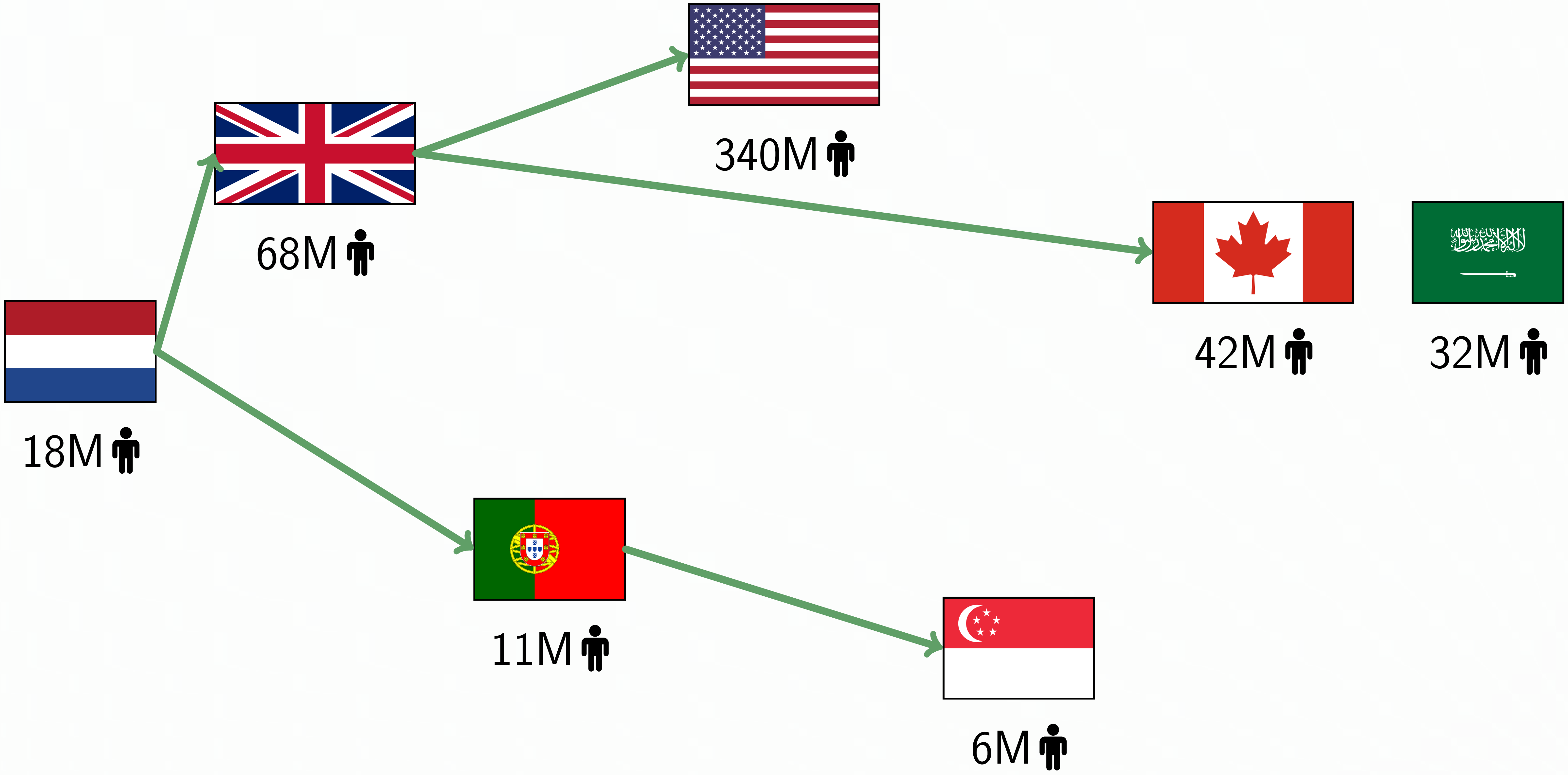

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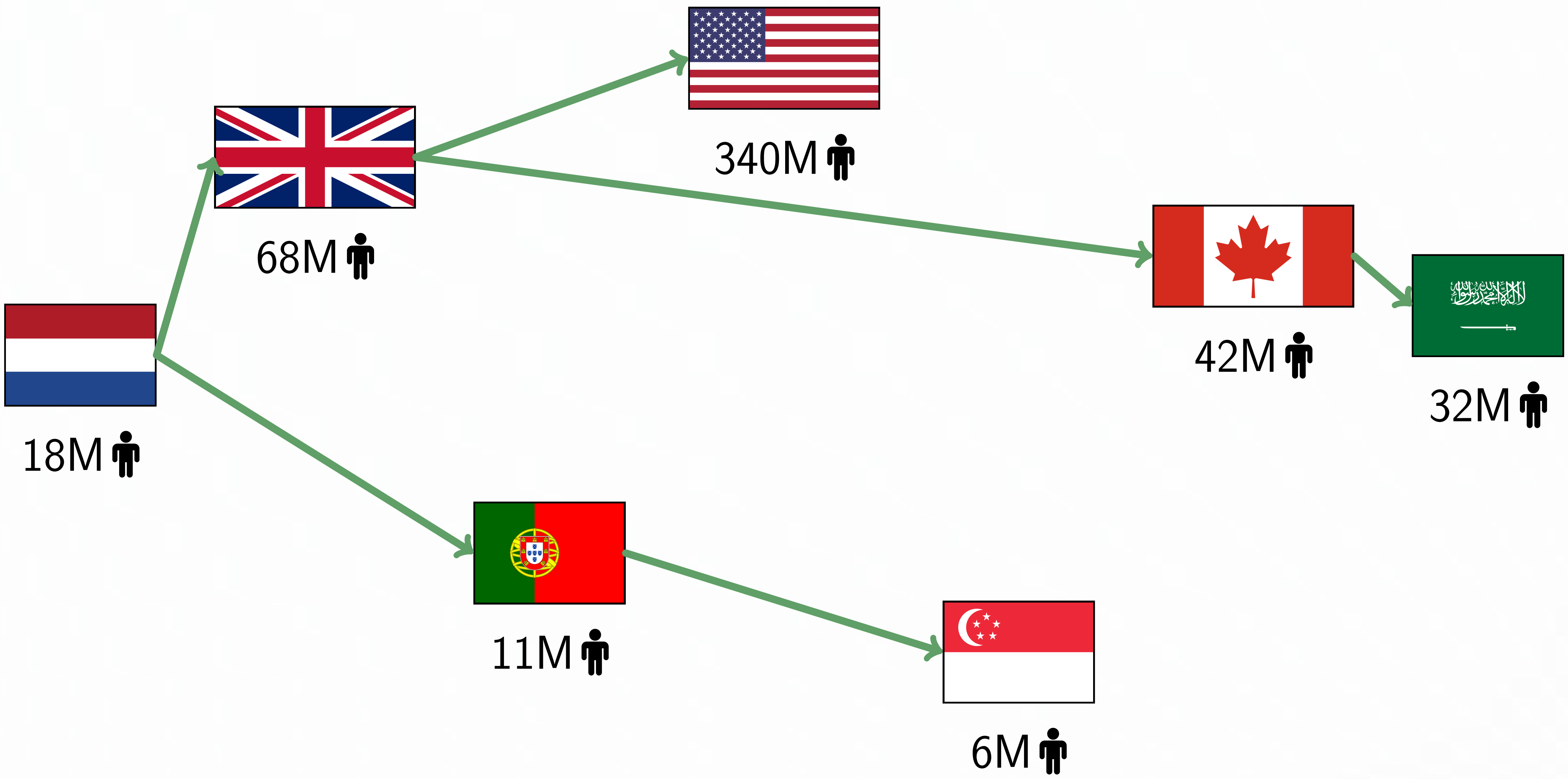

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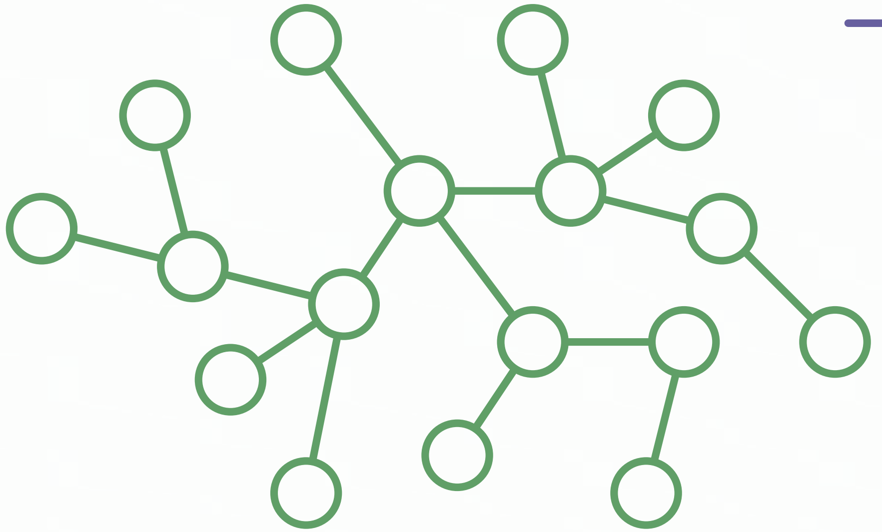

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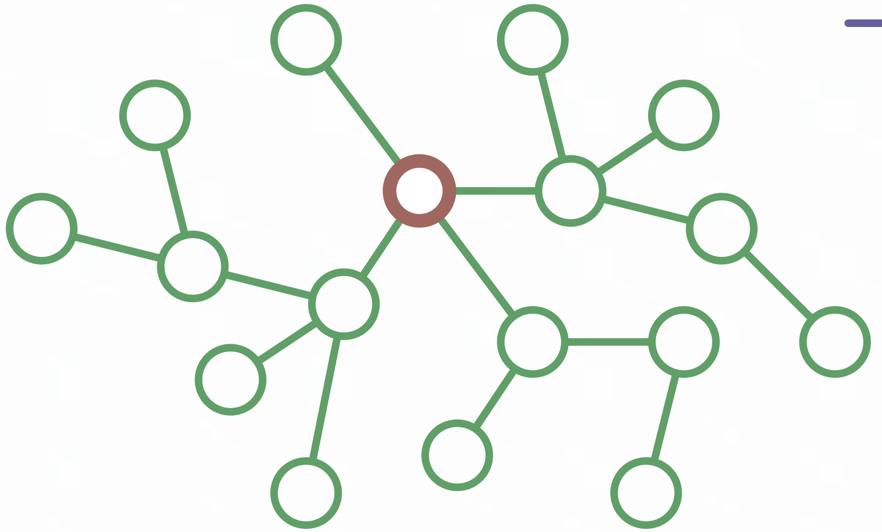
Rooted tree:



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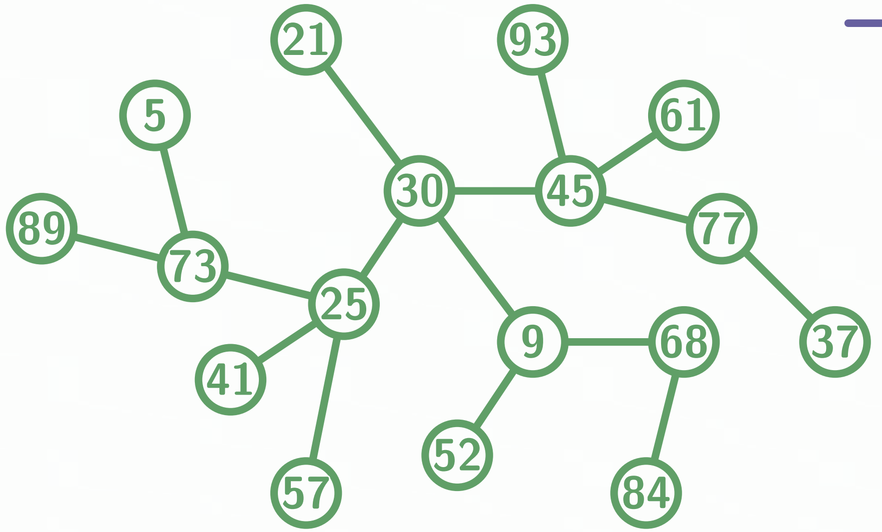
Labelled tree:



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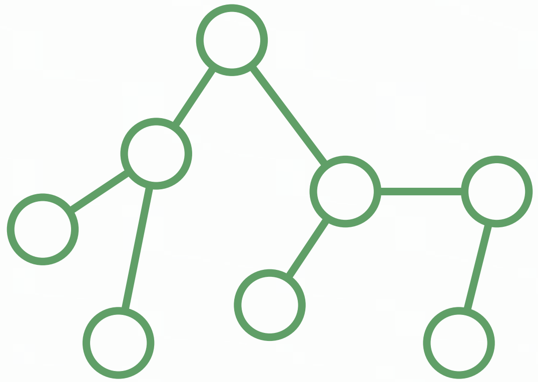

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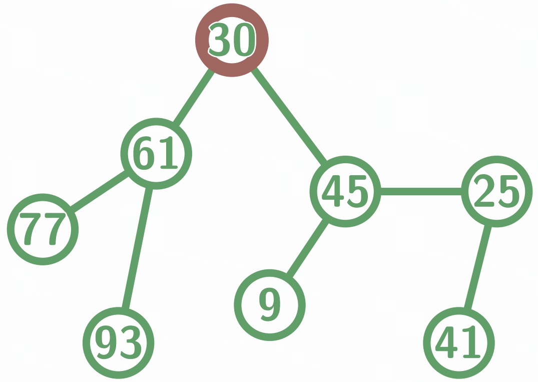

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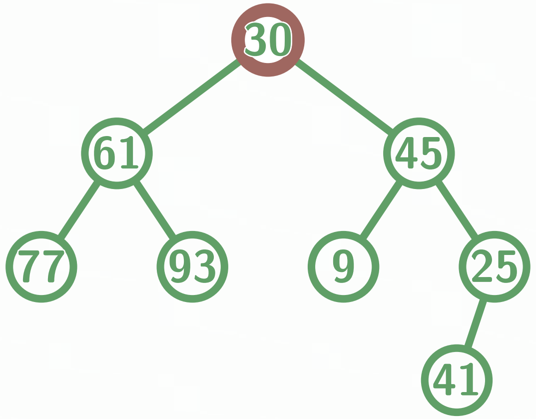

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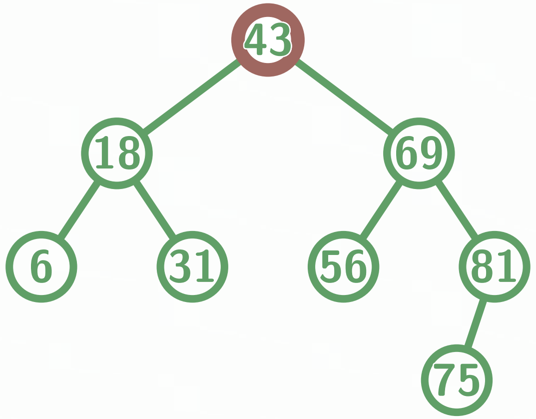

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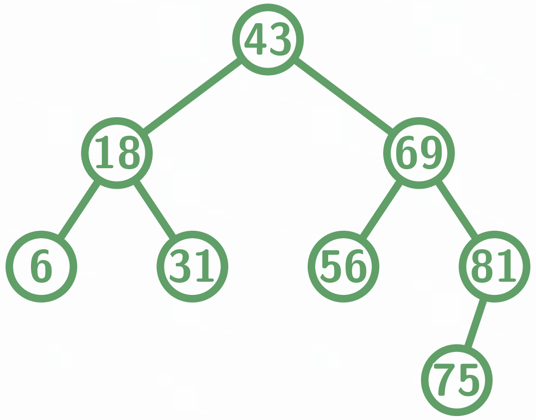

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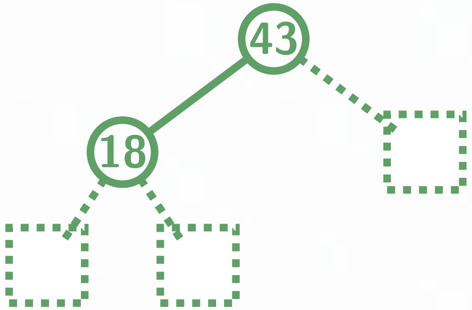

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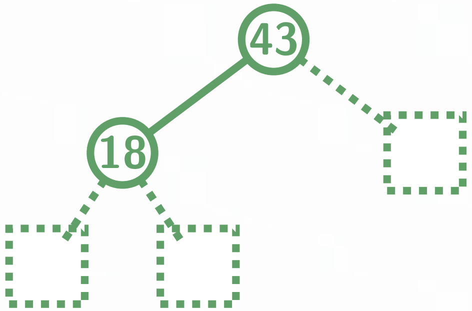

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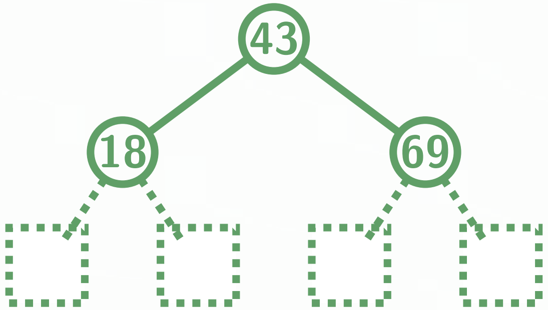

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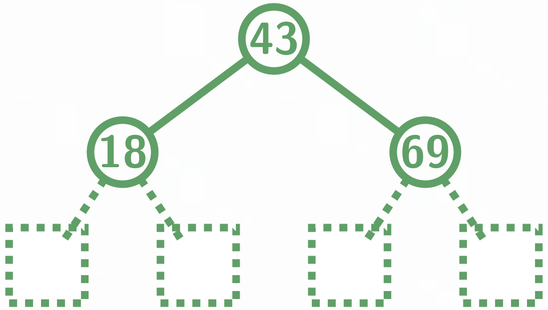

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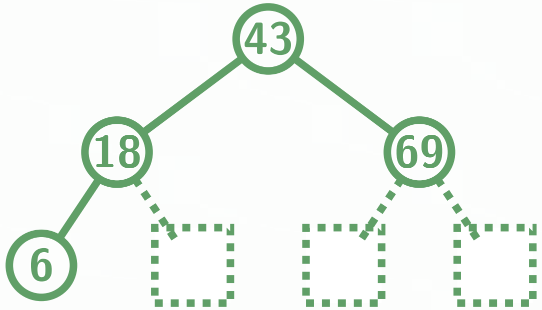

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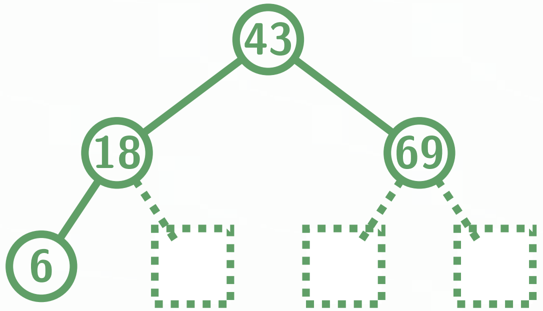

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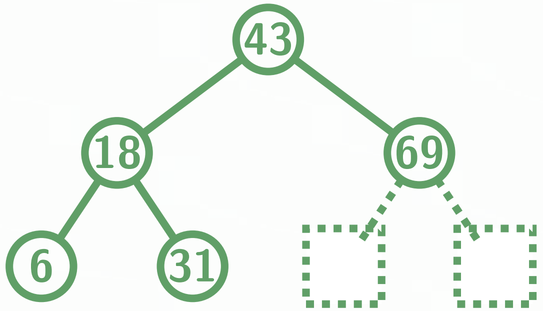

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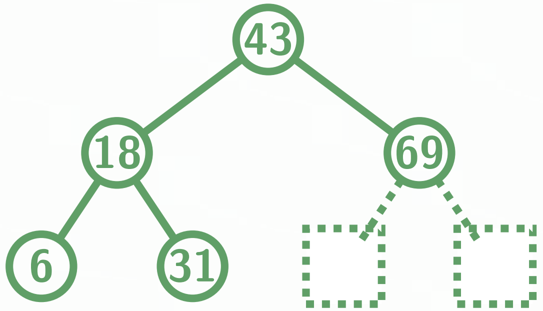

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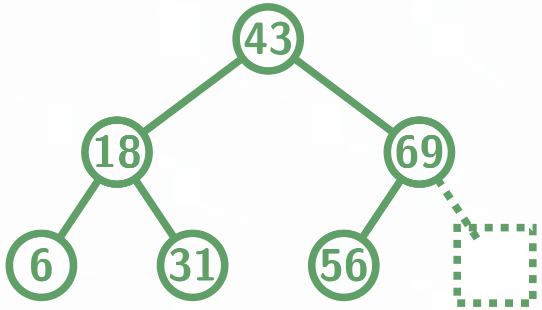

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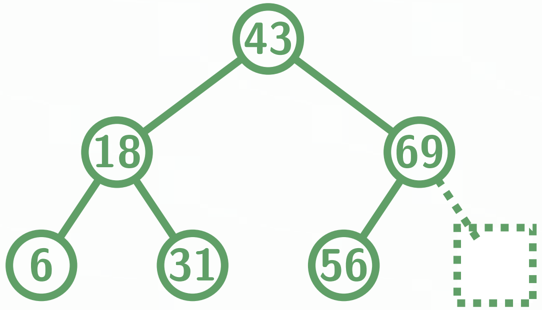

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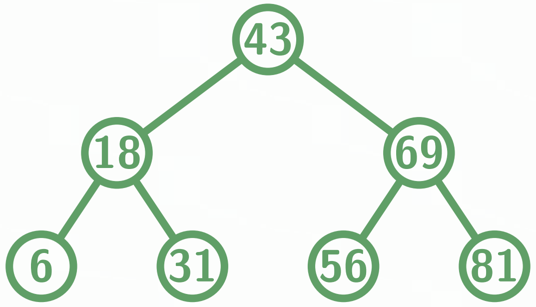

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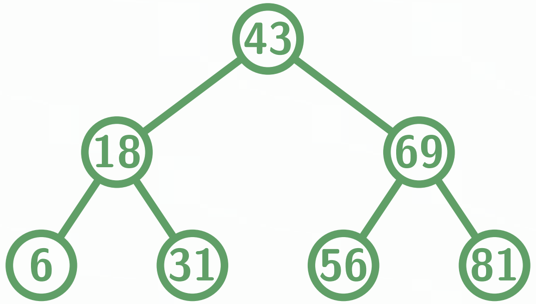

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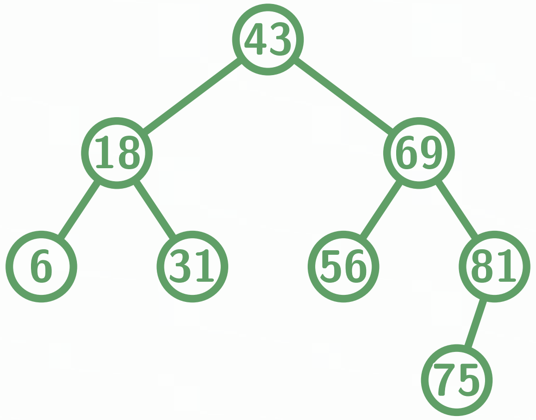

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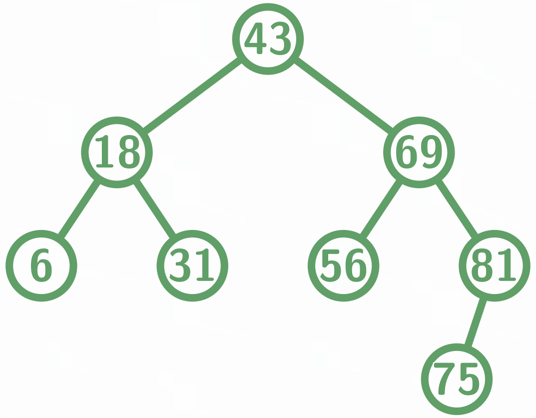

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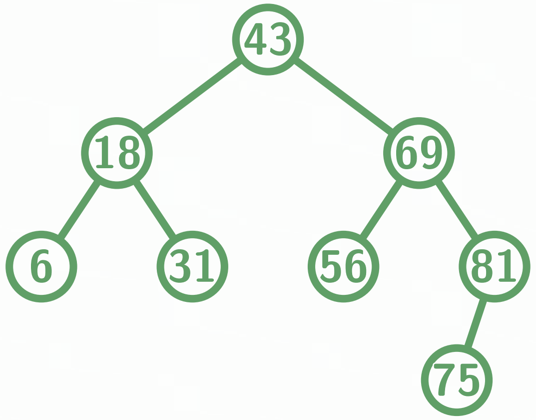

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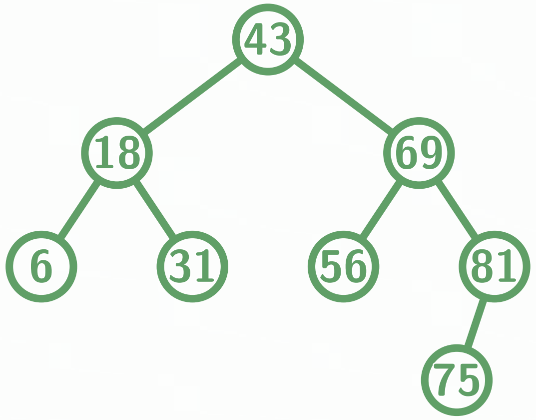

Binary Search Trees

For any sequence of distinct numbers $(x_1, \ldots, x_n)$, there exists a unique binary search tree obtained by recursively inserting the values of the sequence in order.

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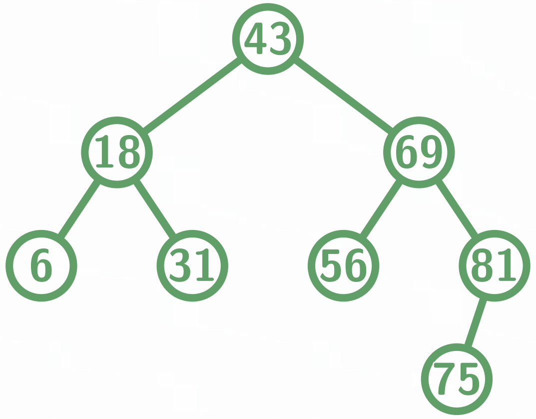

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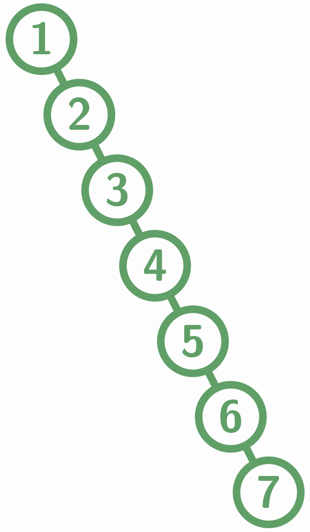

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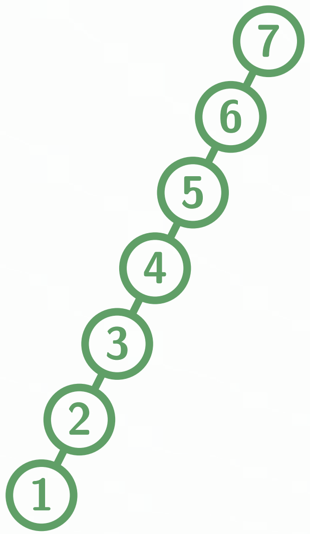

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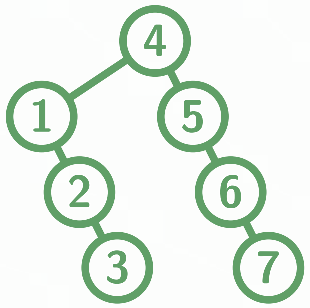

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→ There are also a relatively "recent" object (early results from the end of the 70[4]) with a lot of interesting research left to pursue!

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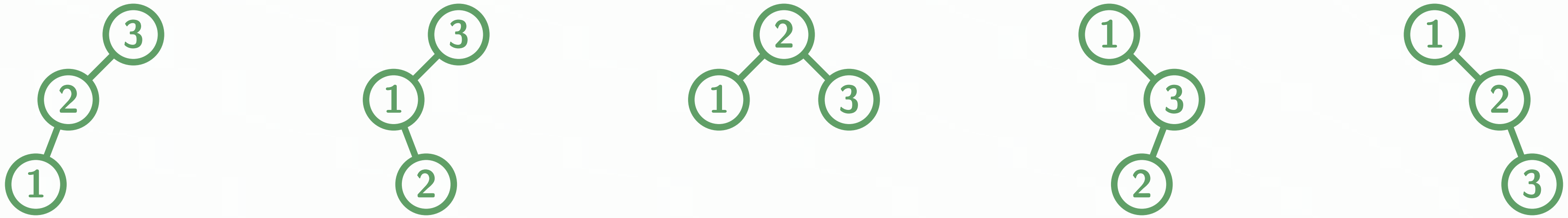

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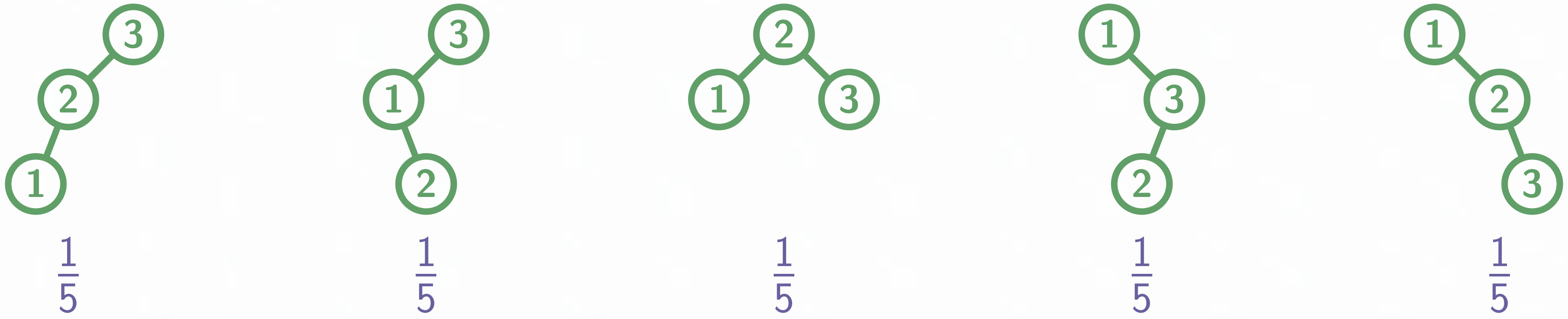

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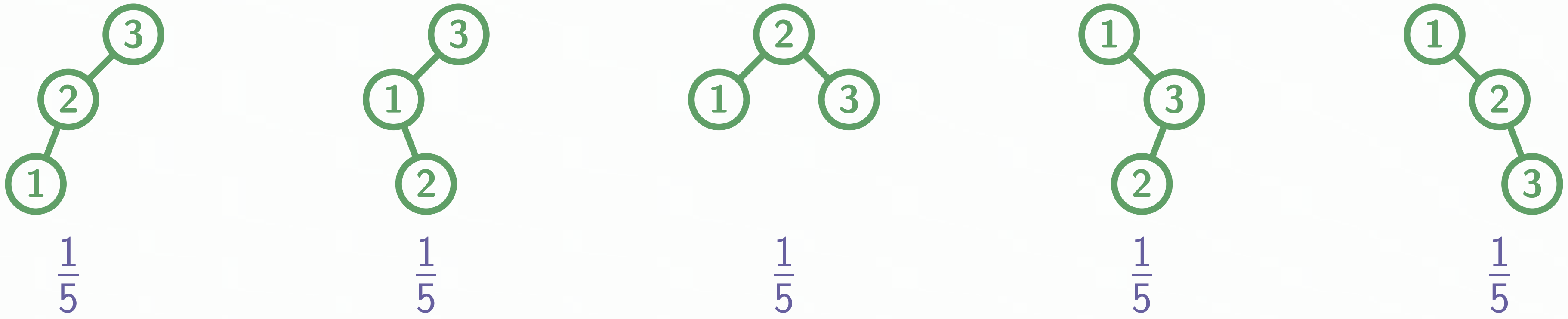

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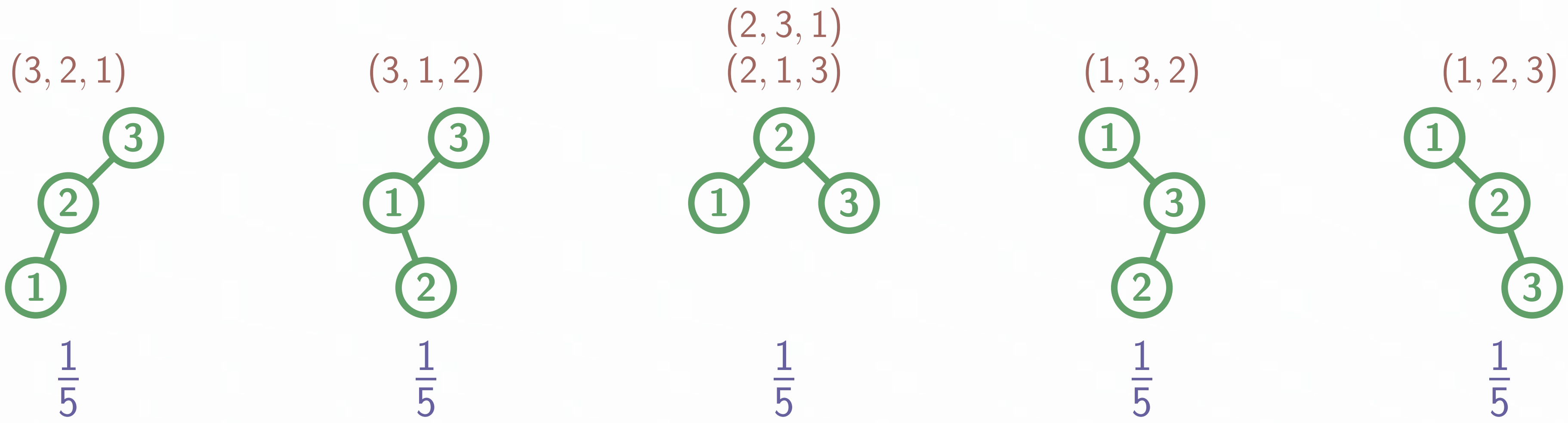

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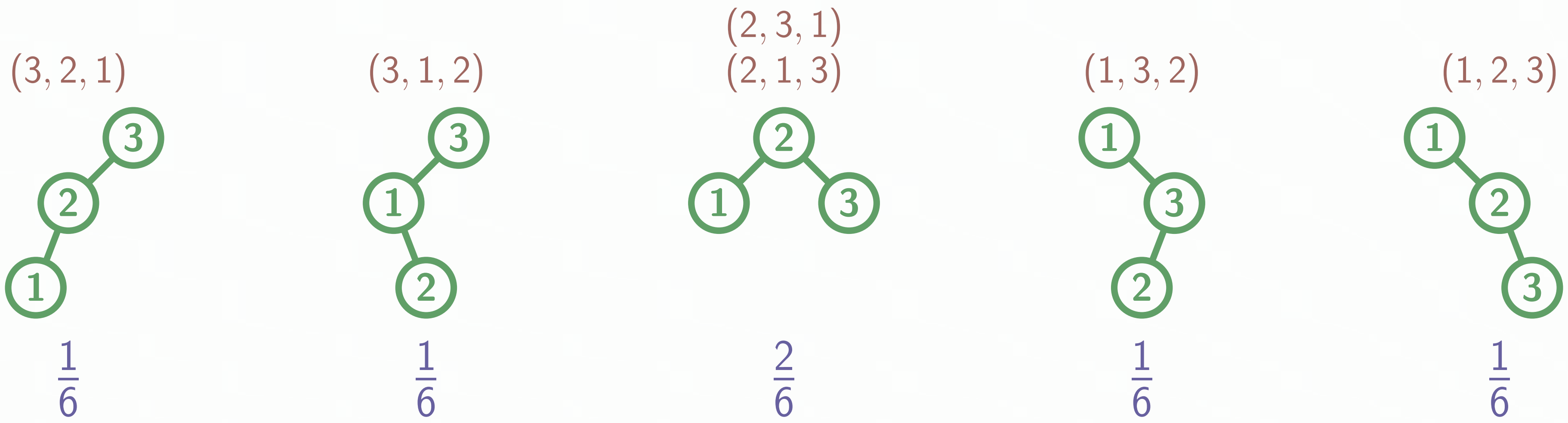

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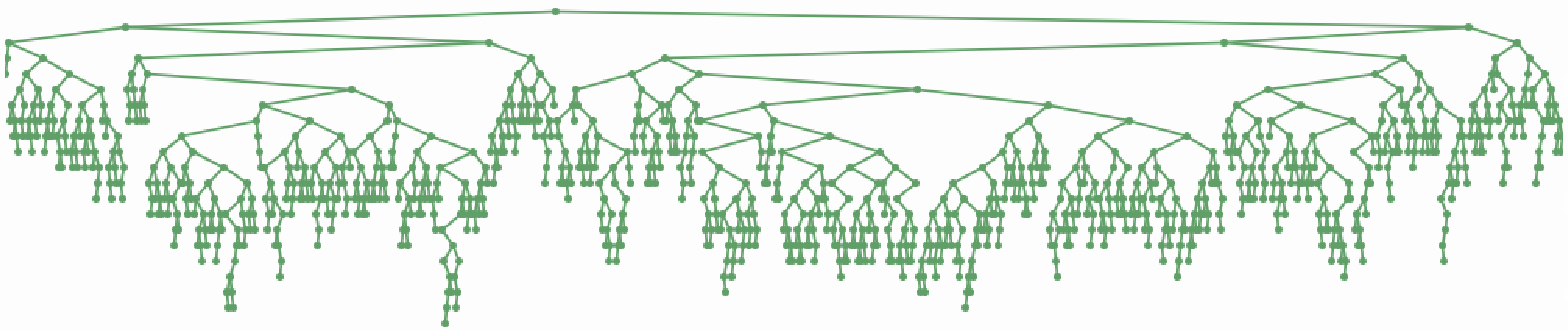

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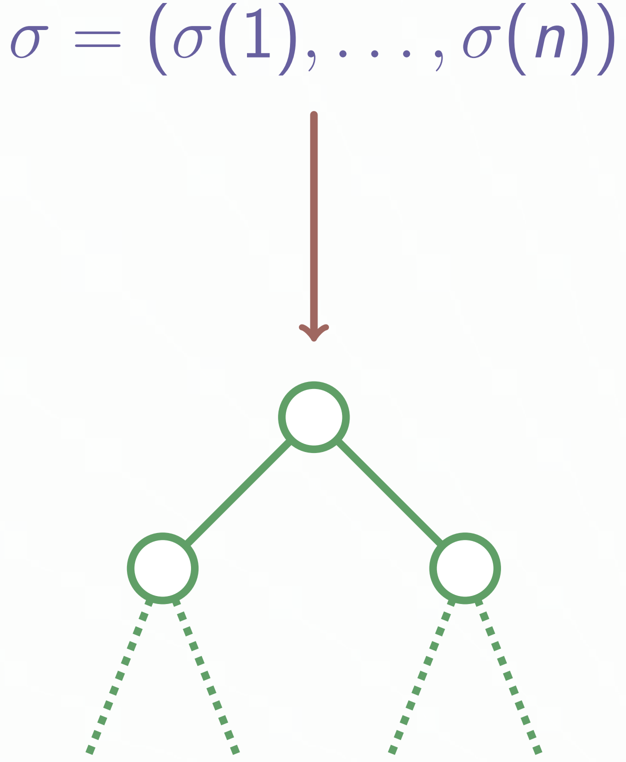

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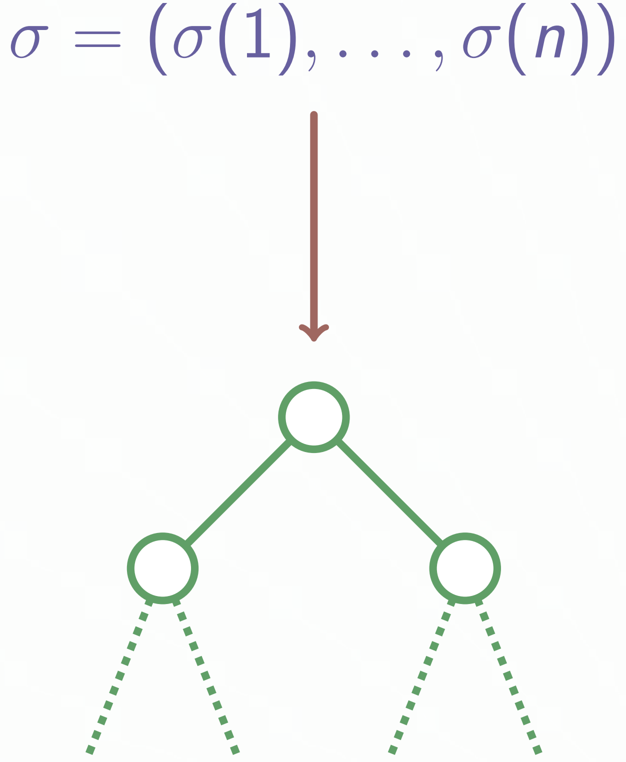

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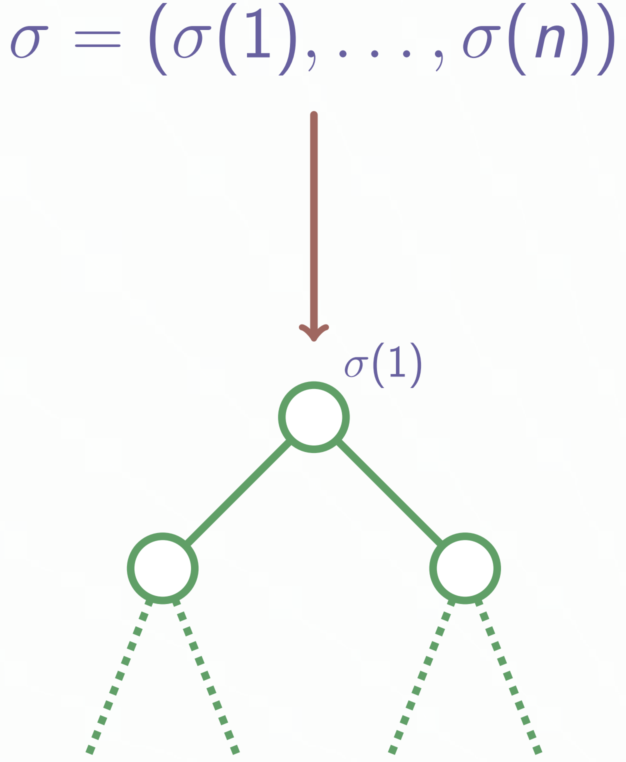

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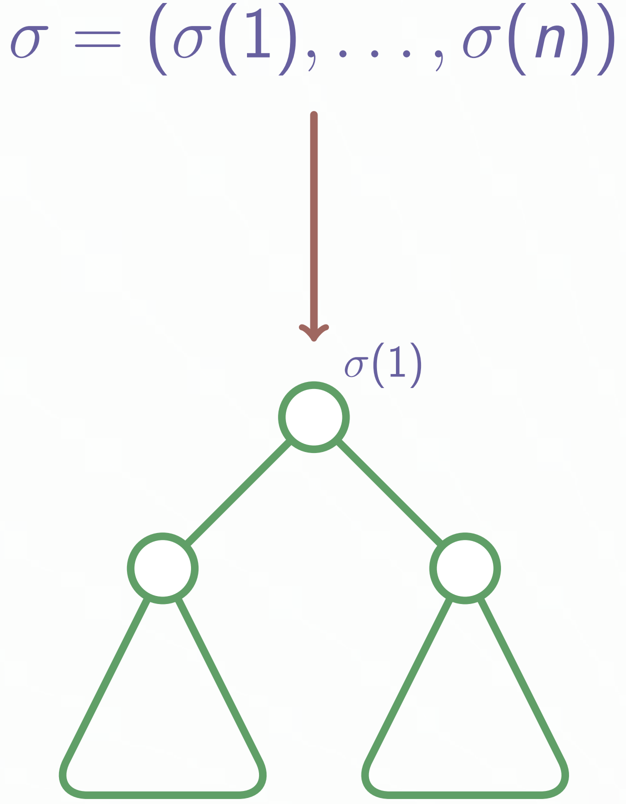

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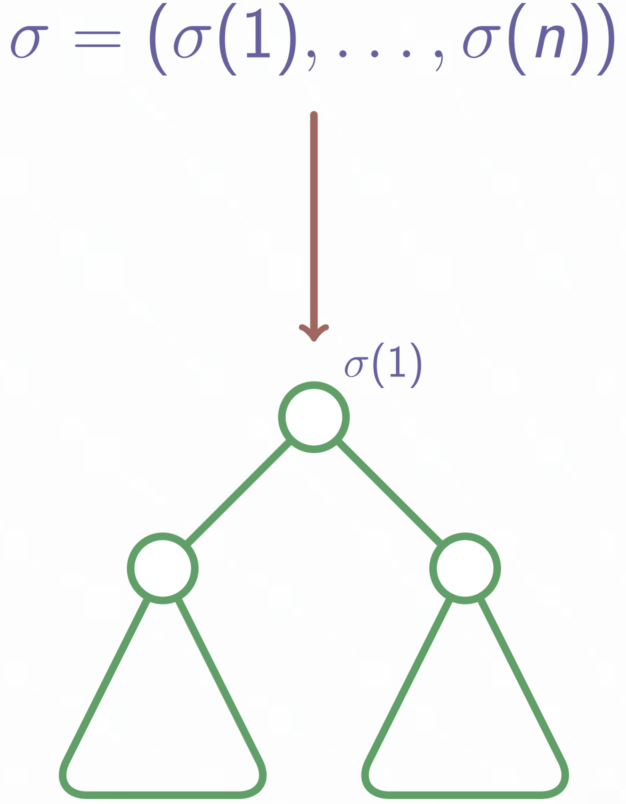

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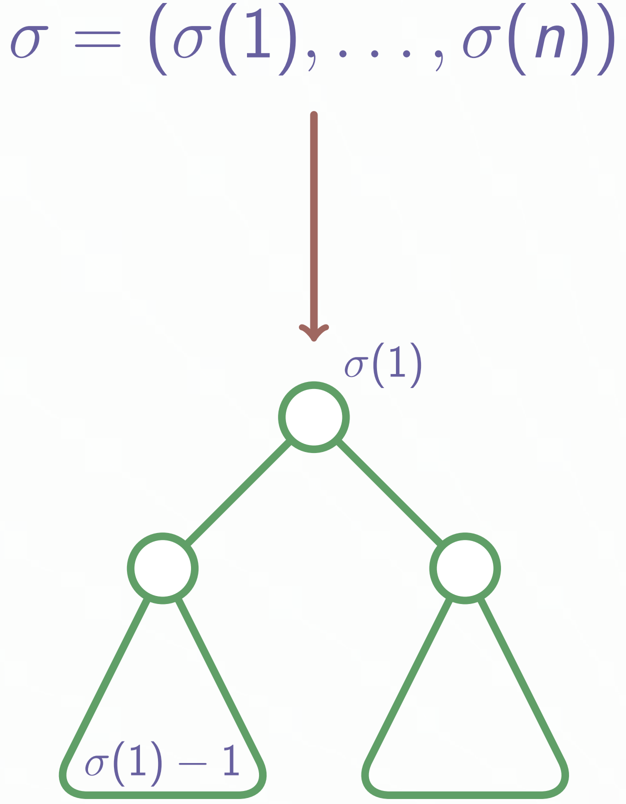

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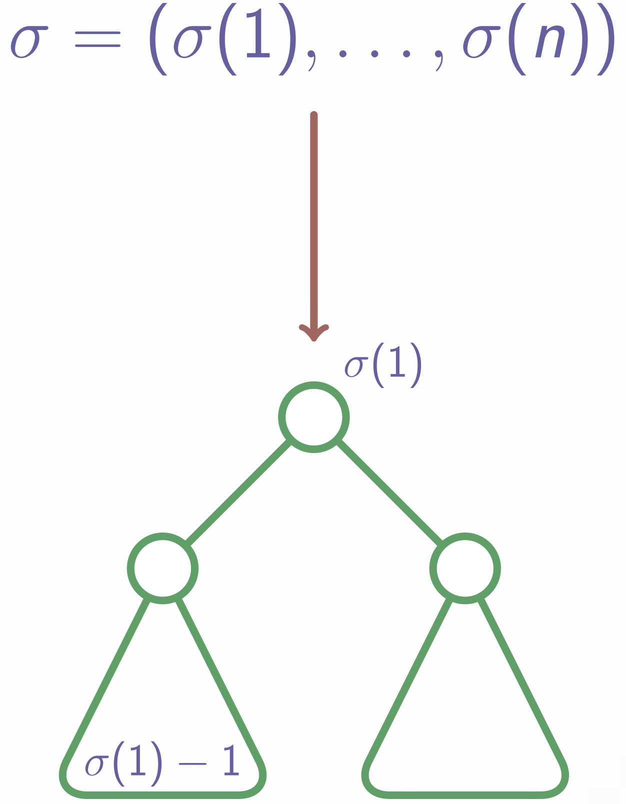

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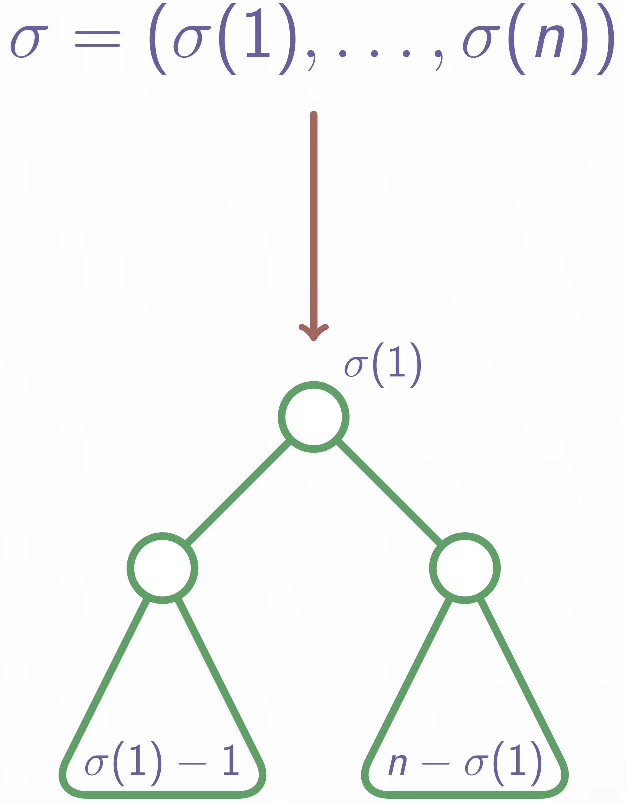

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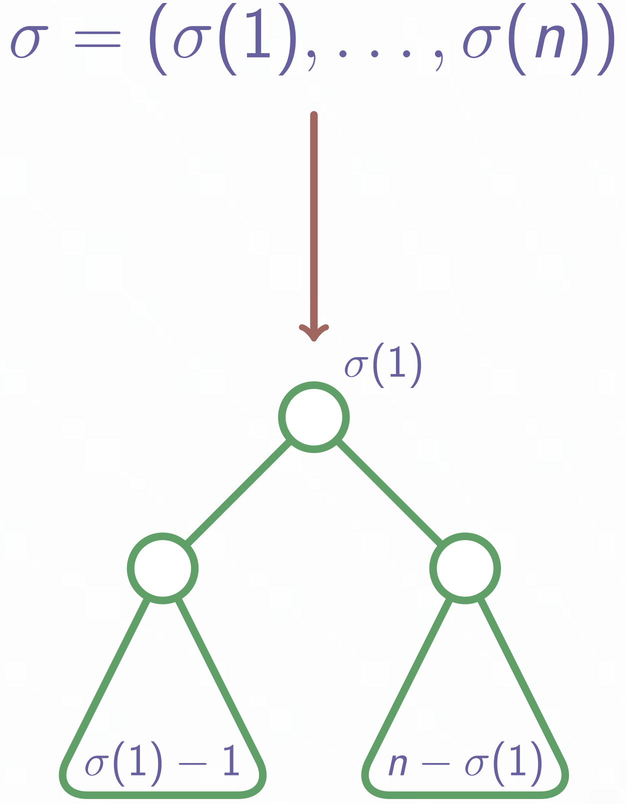

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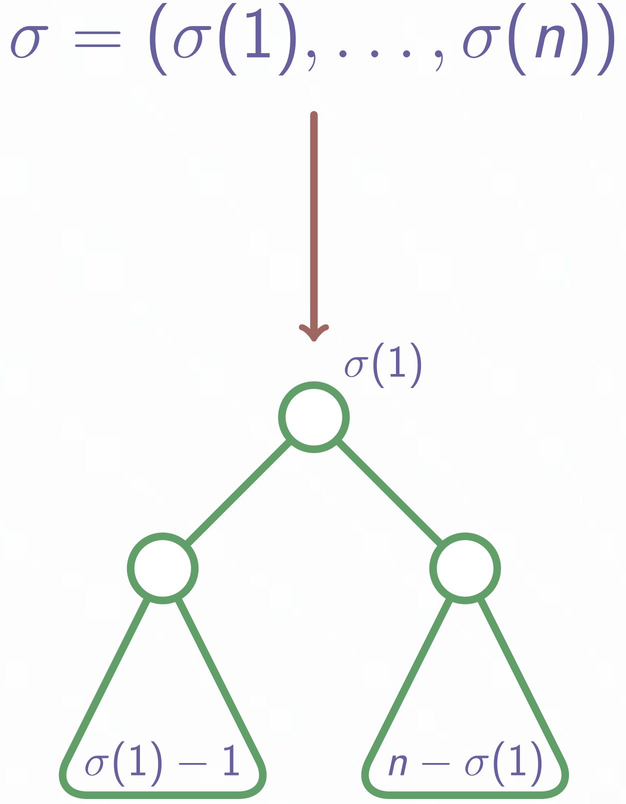

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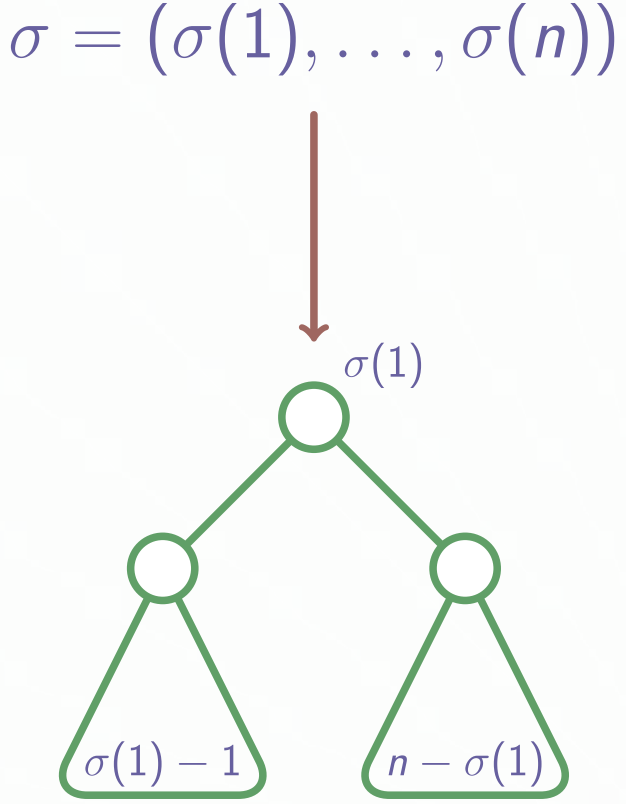

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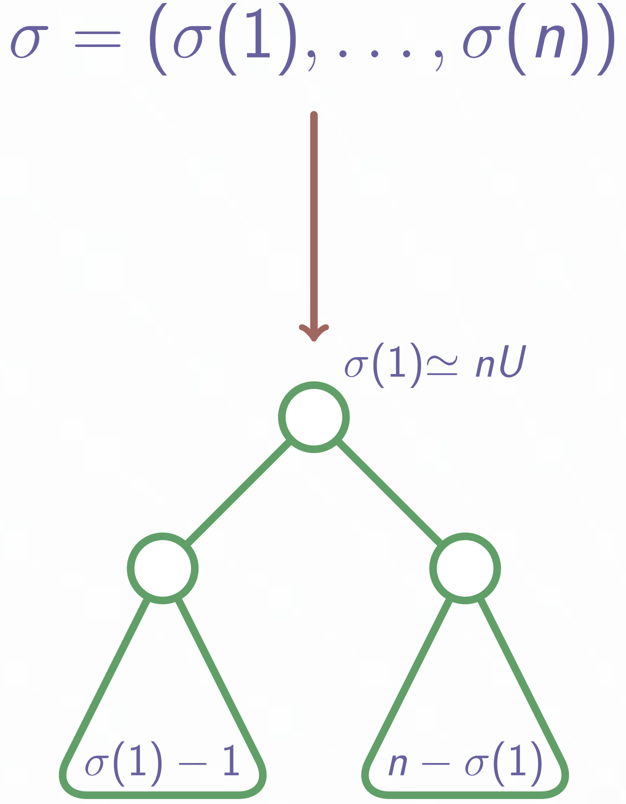

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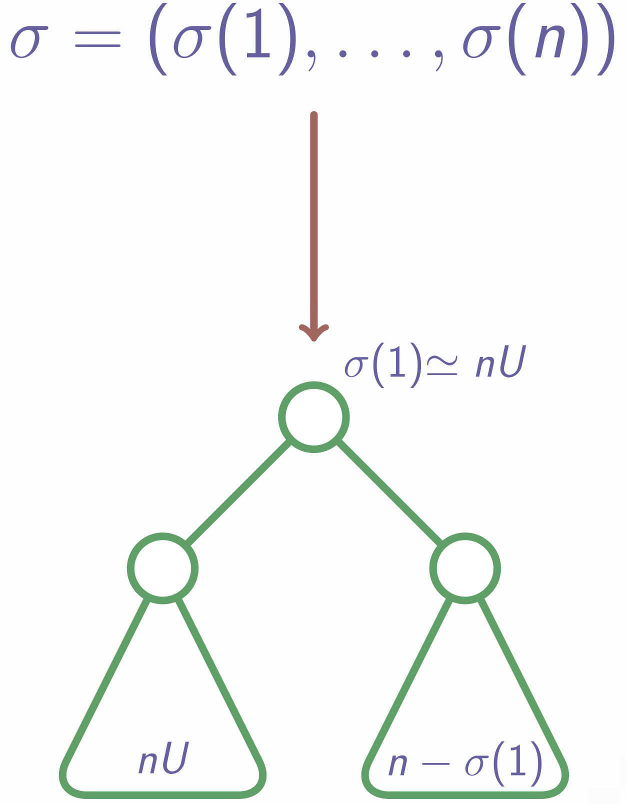

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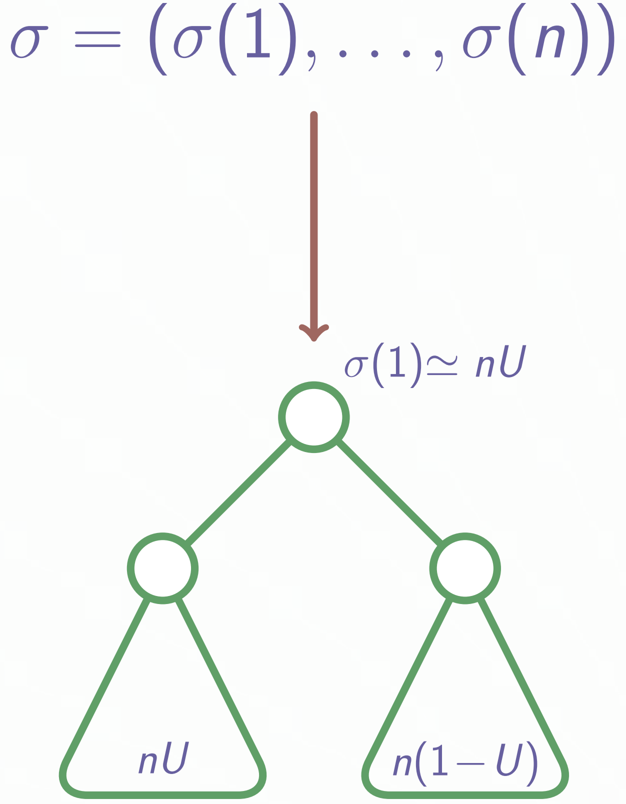

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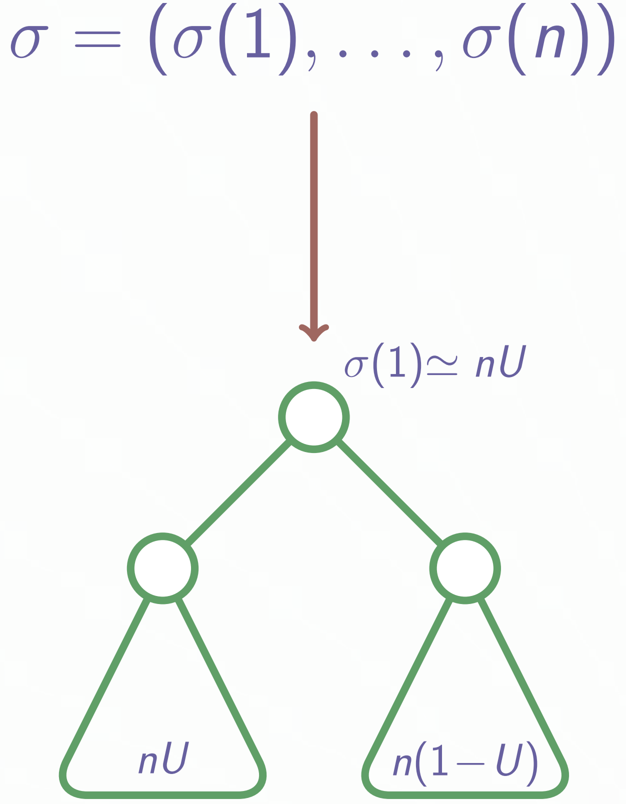

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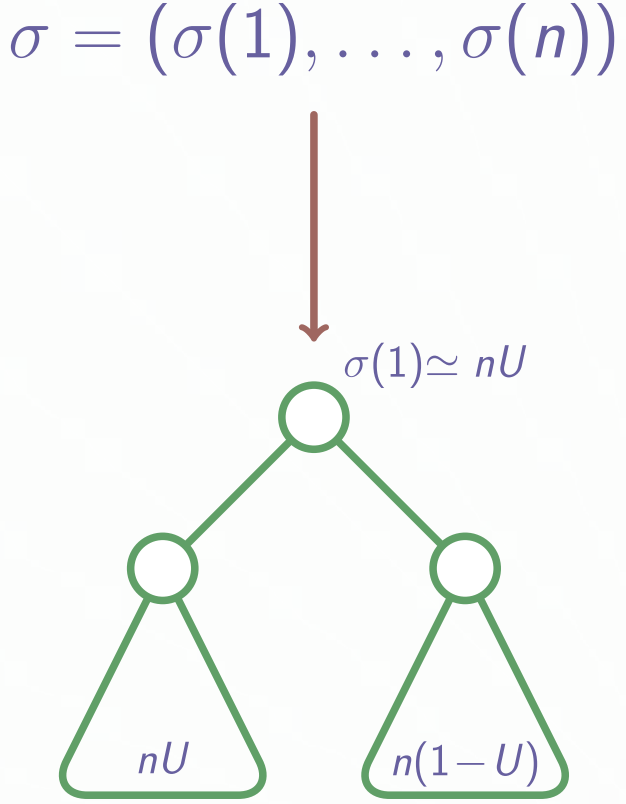

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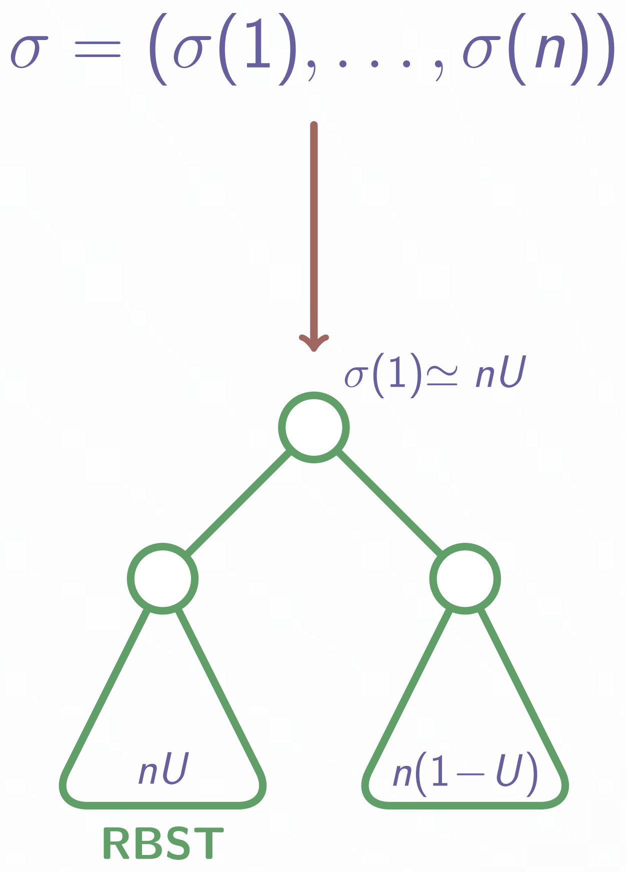

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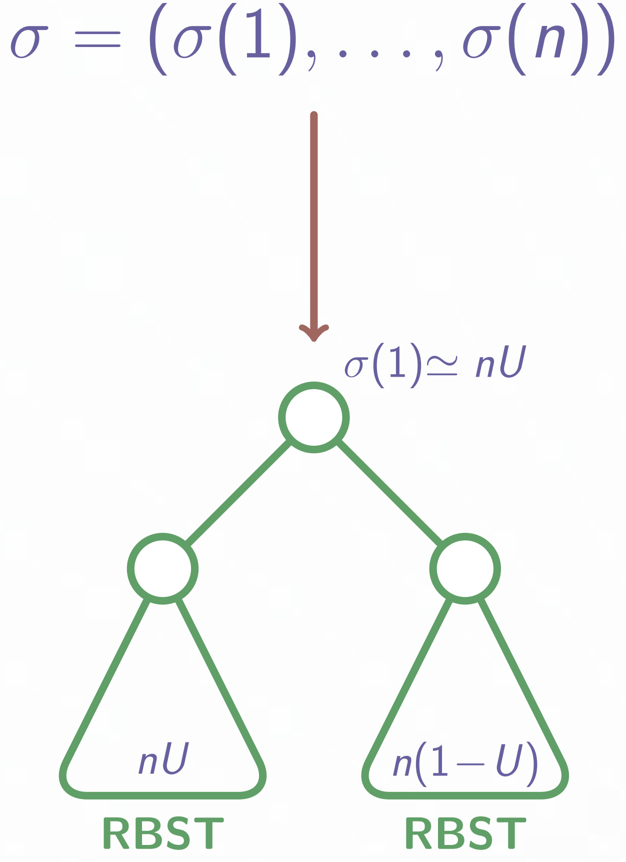

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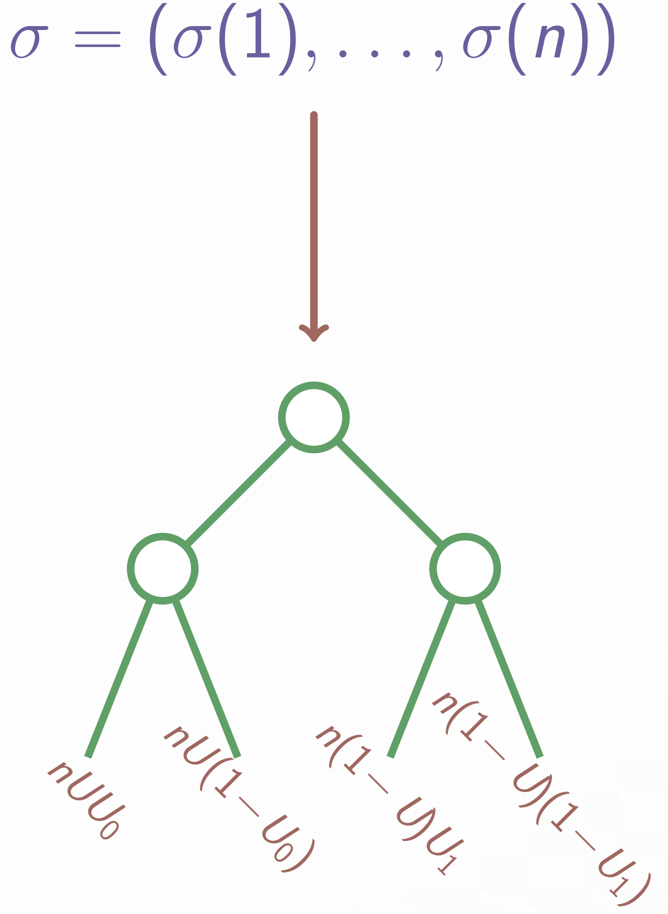

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→ Let us solve this equation.

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We observed that $f(n) = \mathbb{E}[T_n]$ satisfies $f(1) = 1$ and

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- The solution is exactly $f(x) = 1 + 2\log x$, implying that $\mathbb{E}[T_n] = 1 + 2\log n$.

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$\rightarrow$ On average it takes us around $2\log n$ steps to find the correct value!

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- This "splitting tree" approach is quite robust to show other properties of the tree, such as its height[2], corresponding to the worst case scenario.

Table of Contents

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$\longrightarrow$ For each such type can we define corresponding binary search trees?

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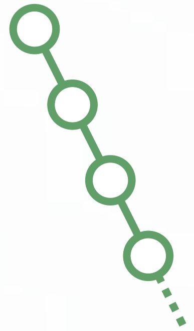

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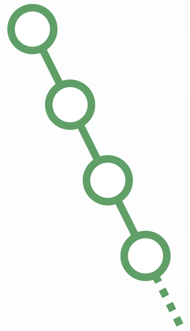

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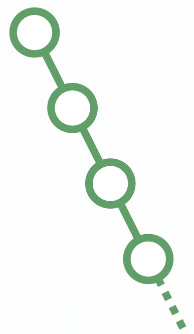

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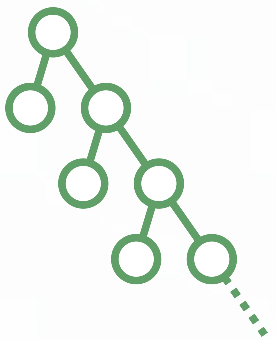

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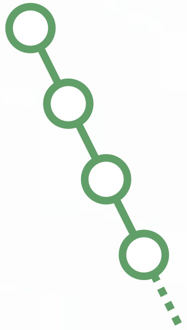

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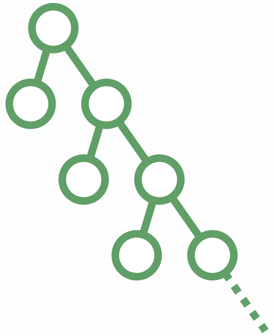

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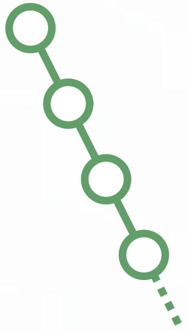

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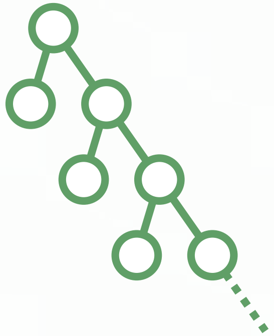

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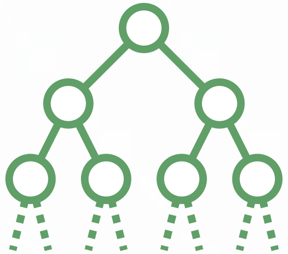

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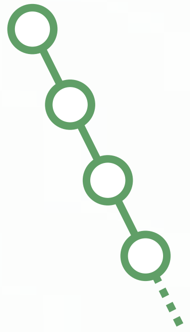

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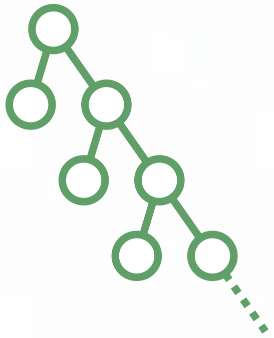

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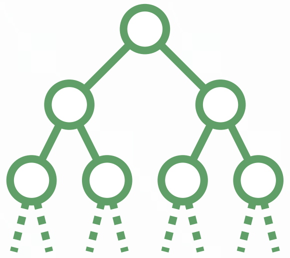

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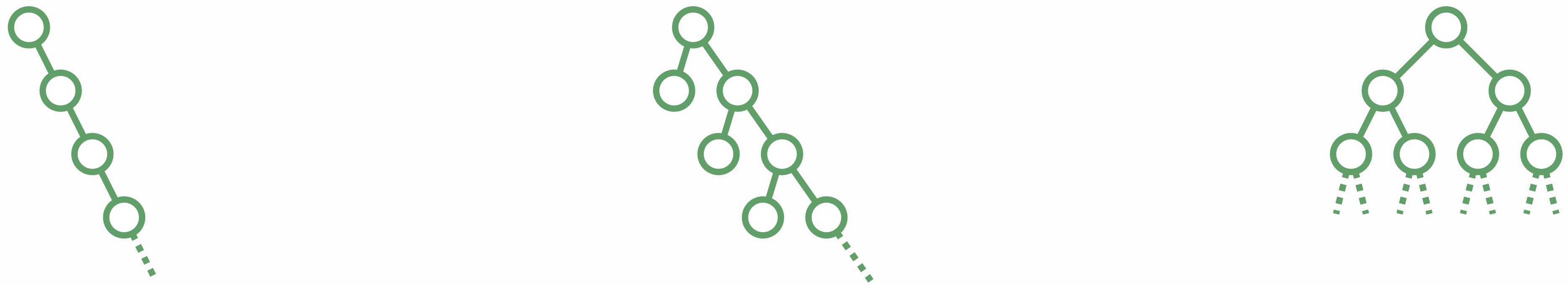

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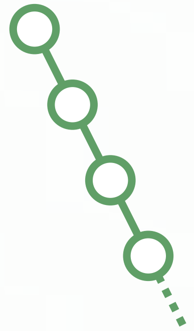

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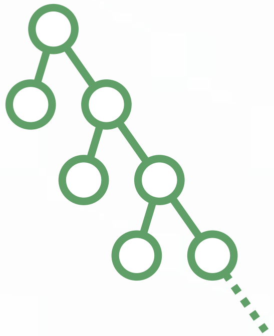

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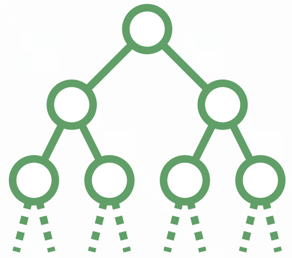

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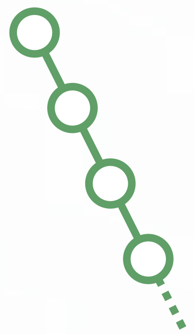

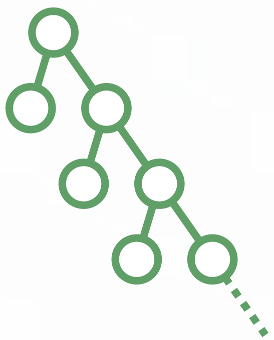

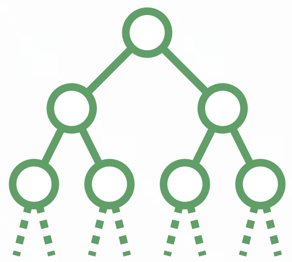

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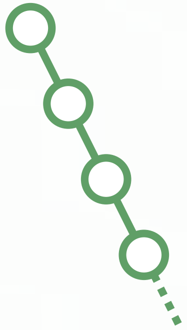

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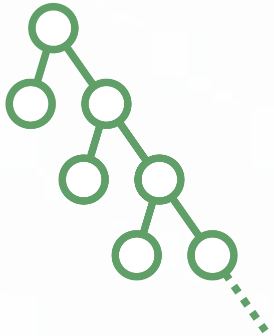

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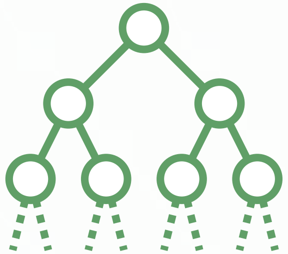

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Let's skip this one for now...

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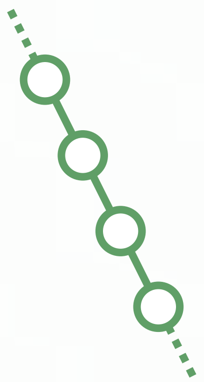

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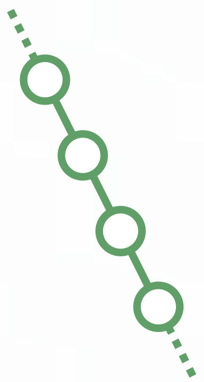

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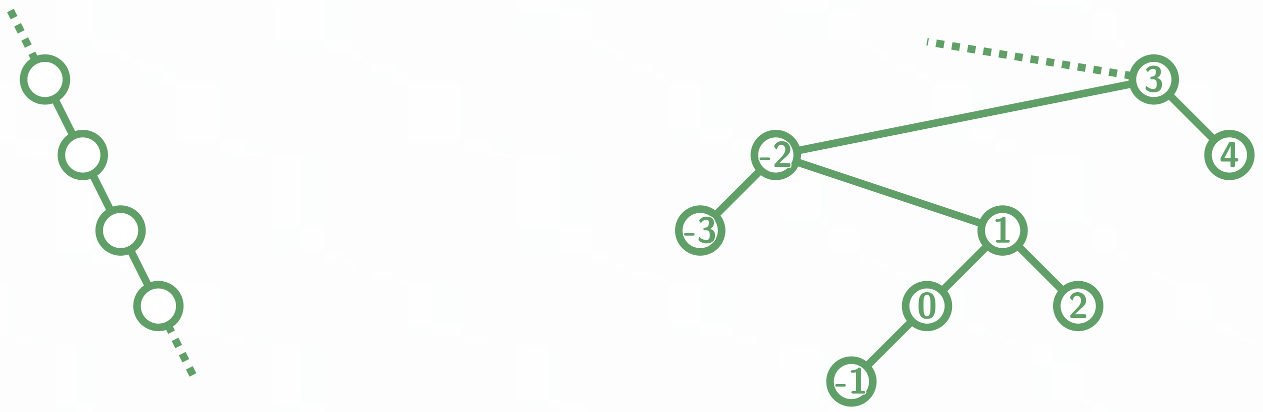

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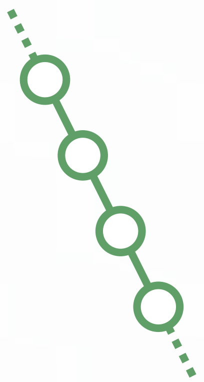

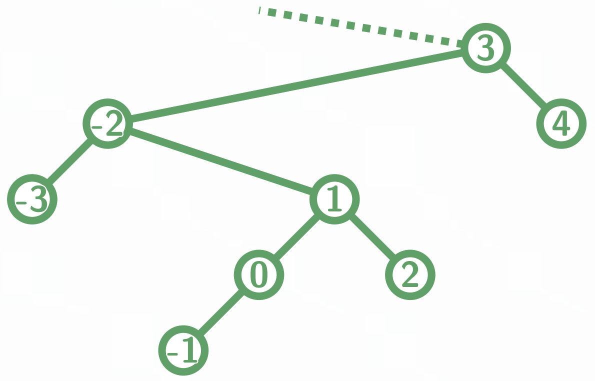

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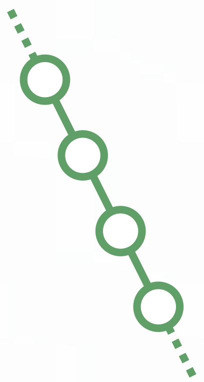

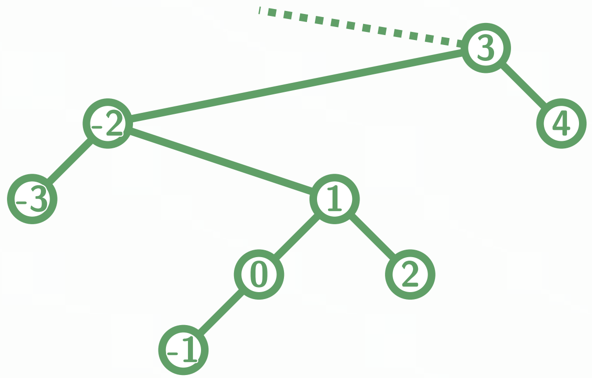

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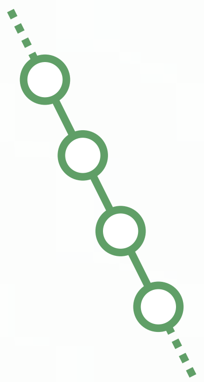

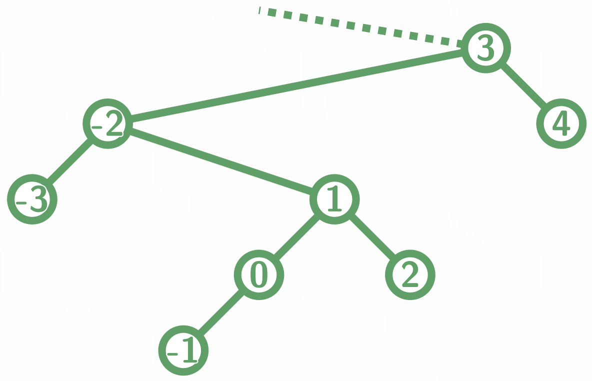

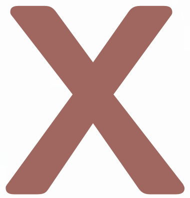

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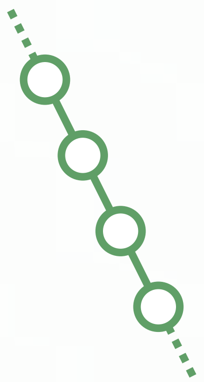

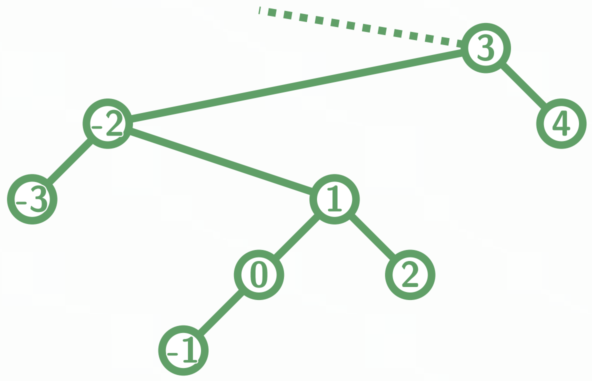

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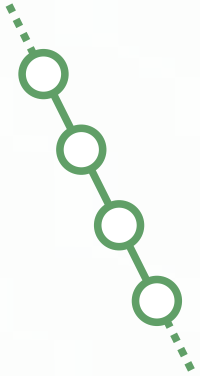

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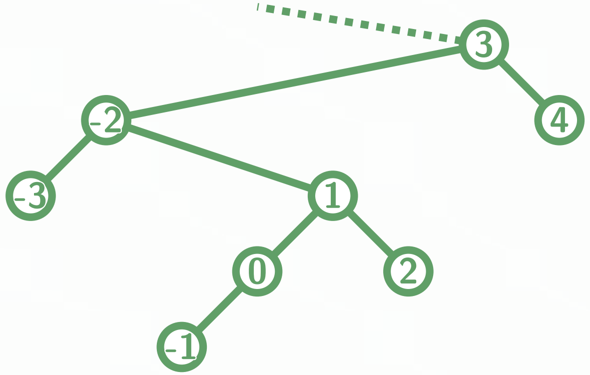

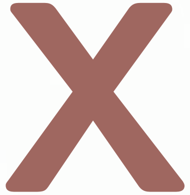

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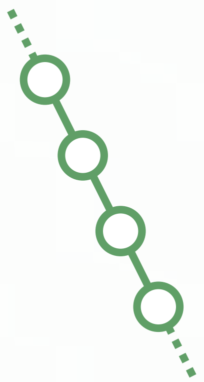

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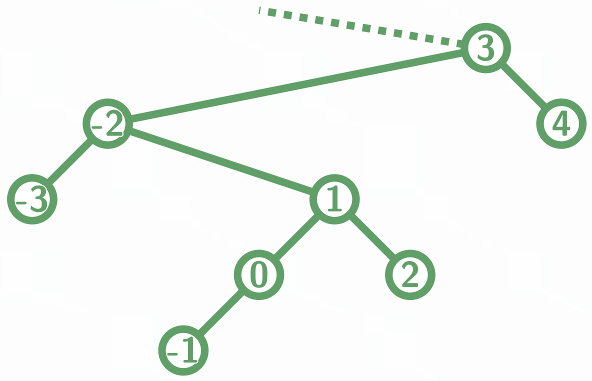

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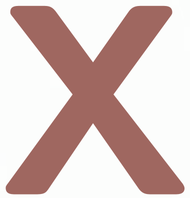

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→ The first question has several answers, which might depend on the application.
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→ The last question is the most important and tends to ignore left-sided sequences.

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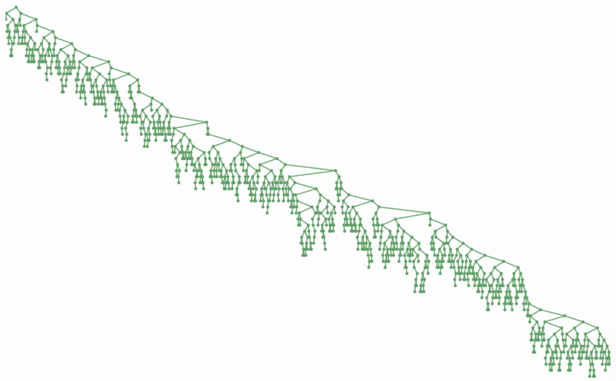

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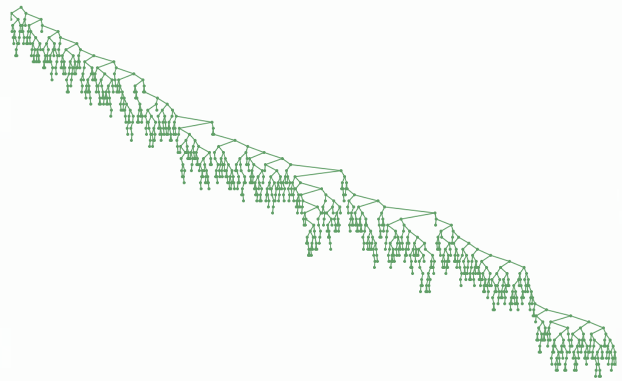

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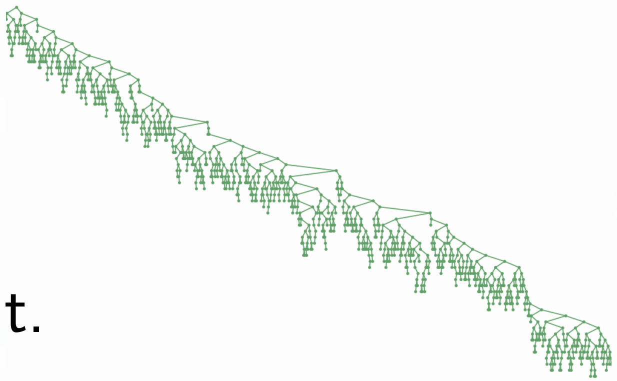

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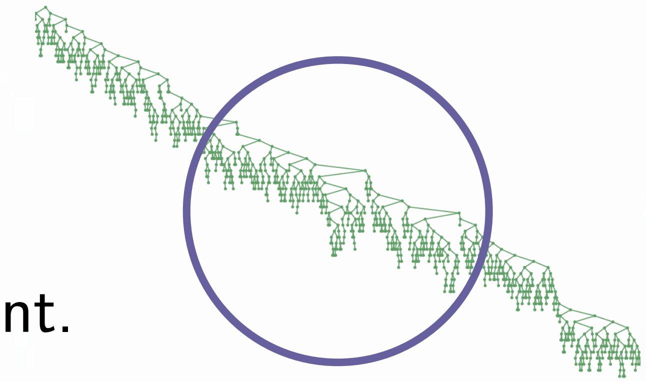

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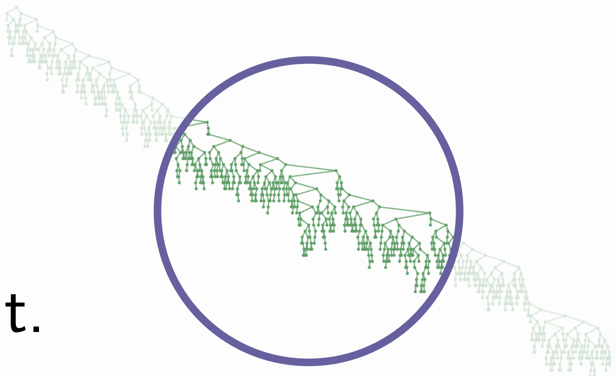

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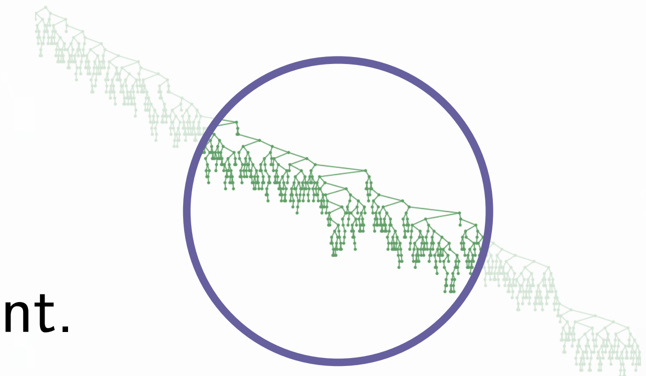

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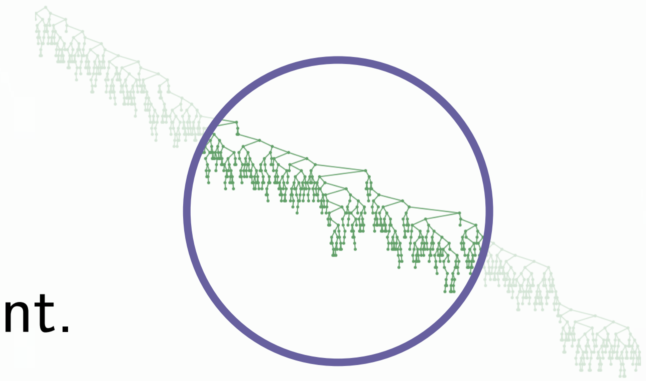

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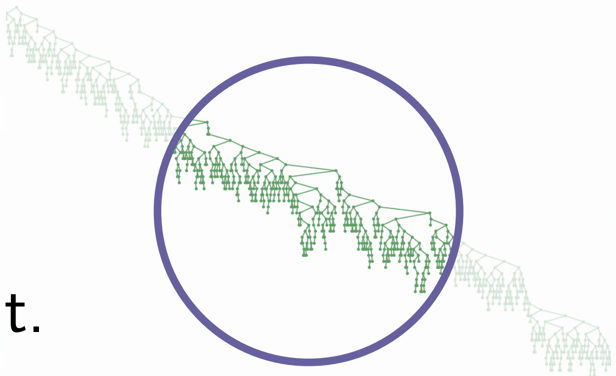

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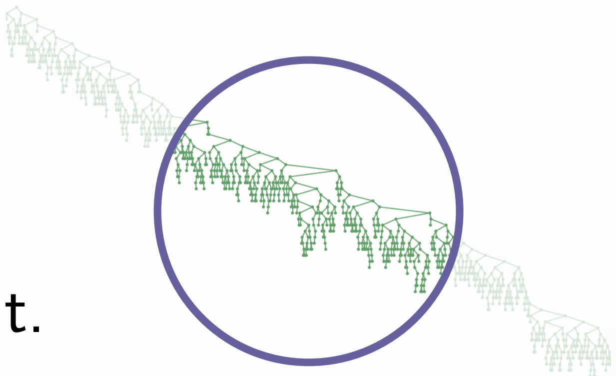

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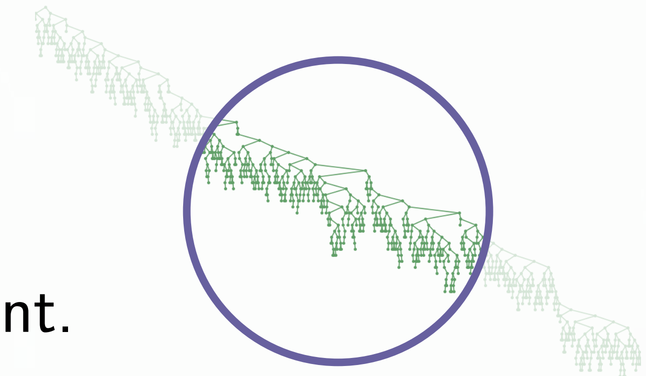

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  - The tree grows up leftward, thus we only need to define 1^{-1}.

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References

1. Corsini, B. (2024). **Limits of Mallows trees**. *Electronic Journal of Probability, 29, 1-44.*.
2. Devroye, L. (1986). **A note on the height of binary search trees**. *Journal of the ACM (JACM), 33(3), 489-498*.
3. Gnedin, A., & Olshanski, G. (2012). **The two-sided infinite extension of the Mallows model for random permutations**. *Advances in Applied Mathematics, 48(5), 615-639*.
4. Robson, J. M. (1979). **The height of binary search trees**. *Australian Computer Journal, 11(4), 151-153*.

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