Local limit of Prim's algorithm

Benoît Corsini

join work with R. Gündlach and R. van der Hofstad

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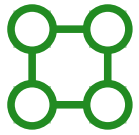

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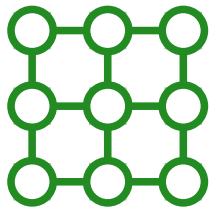

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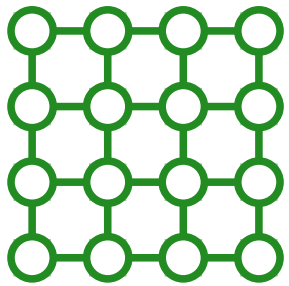

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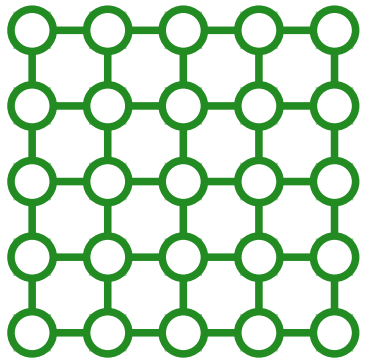

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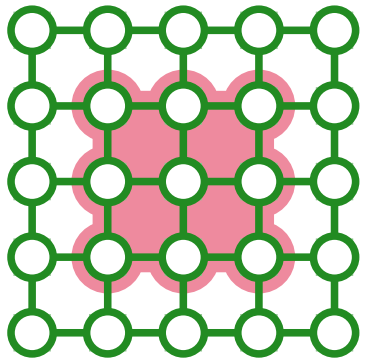

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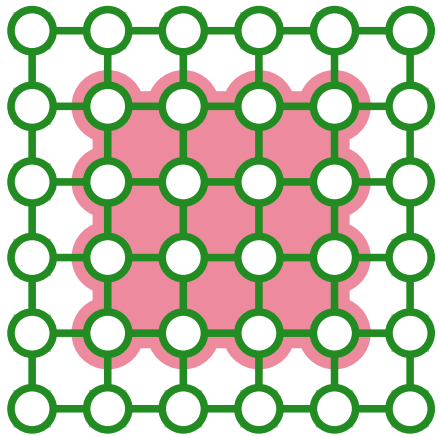

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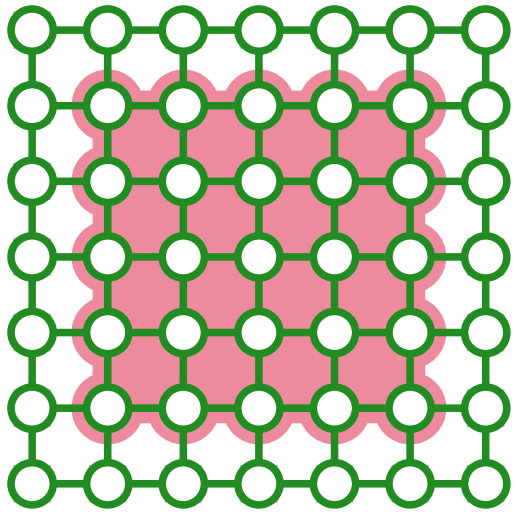

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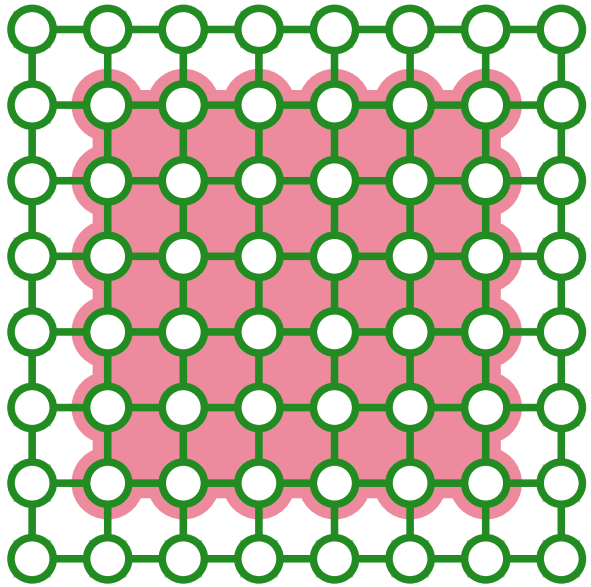

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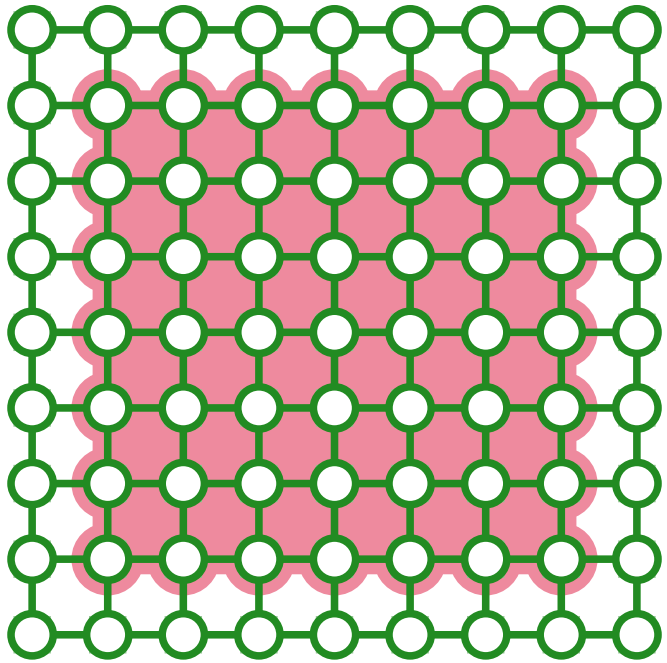

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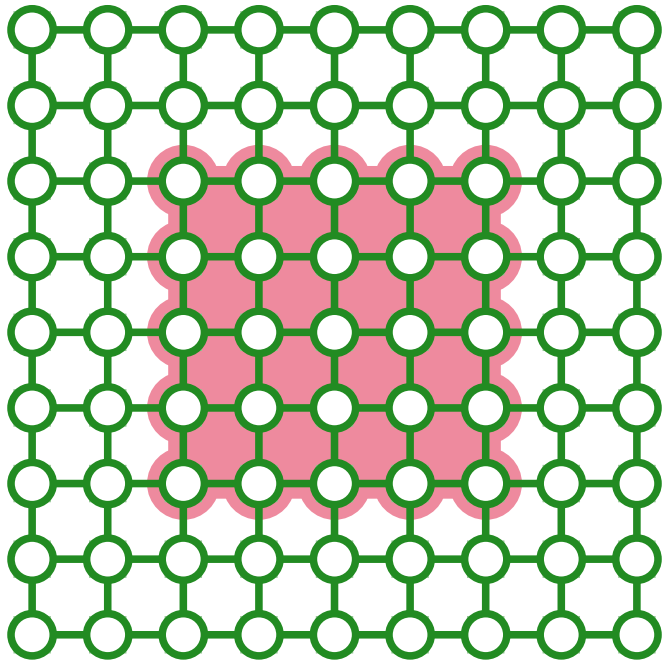

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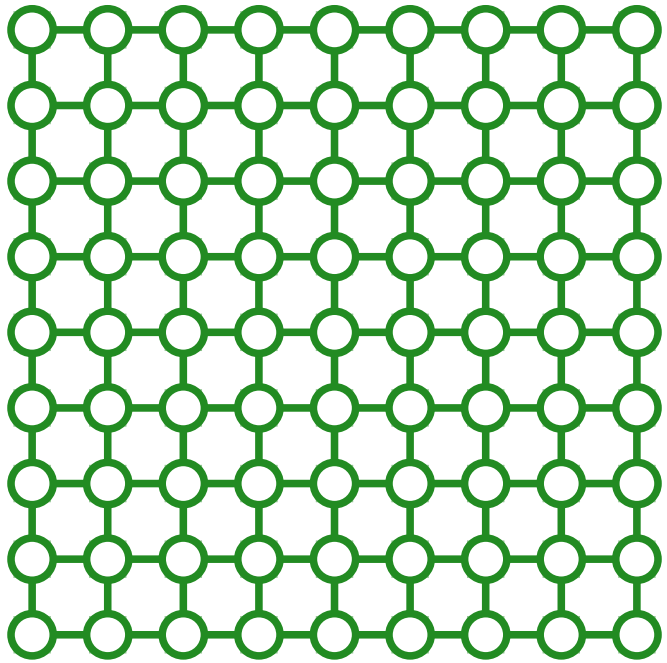

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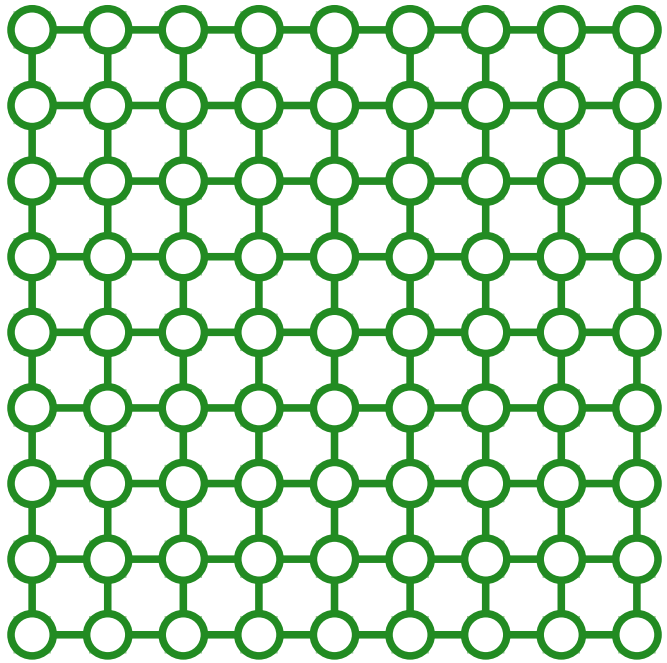

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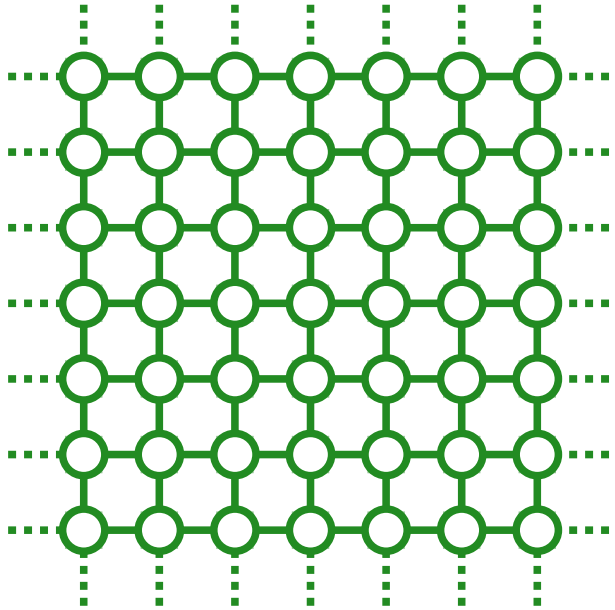

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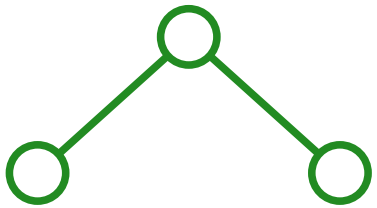

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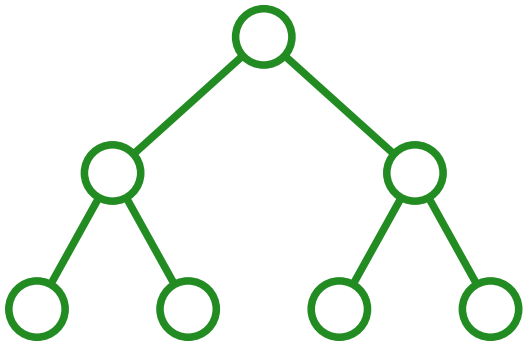

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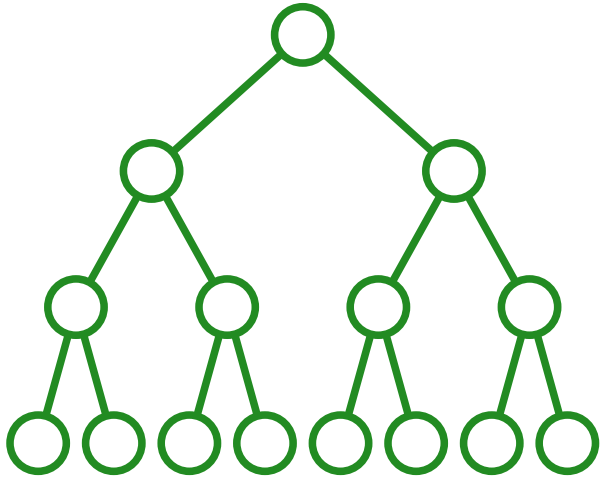

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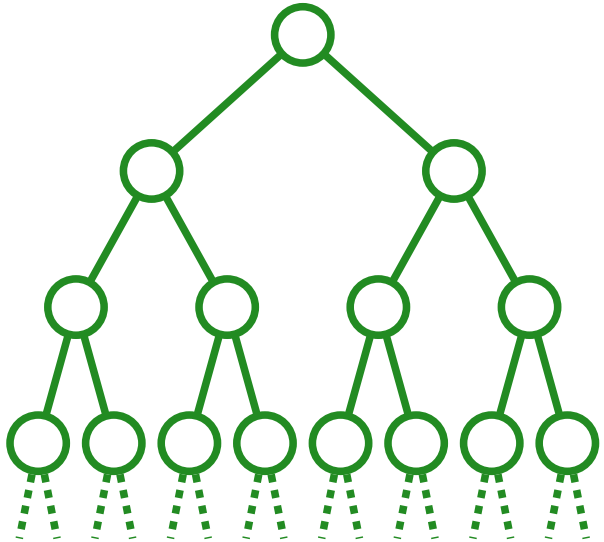

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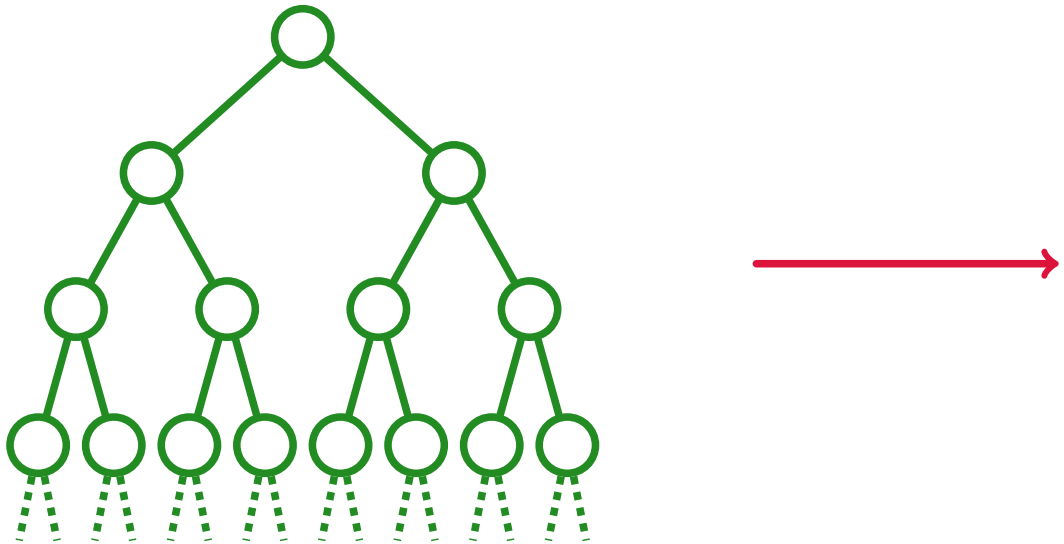

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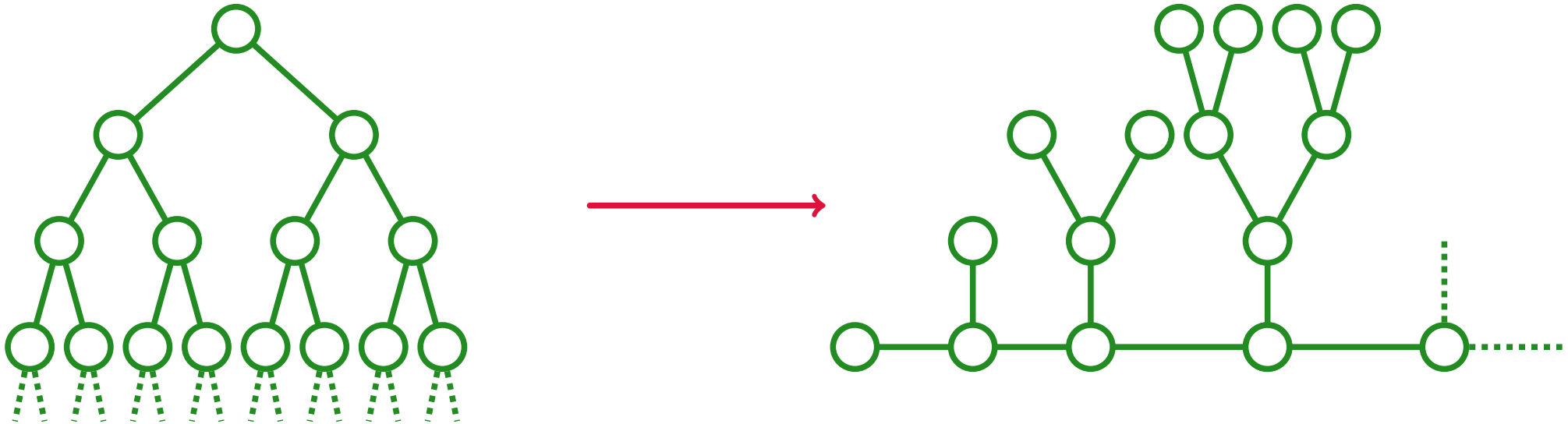

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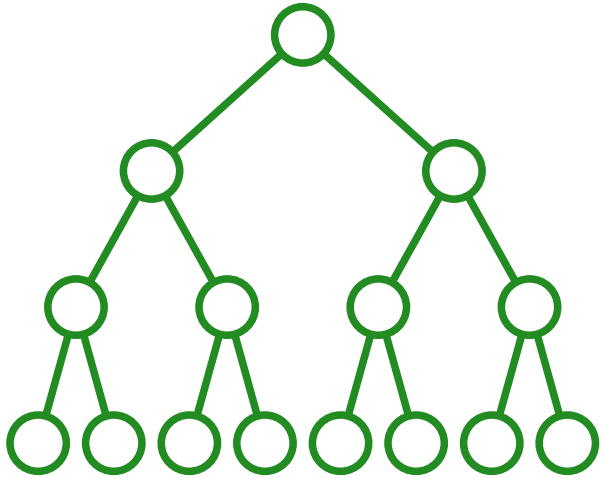

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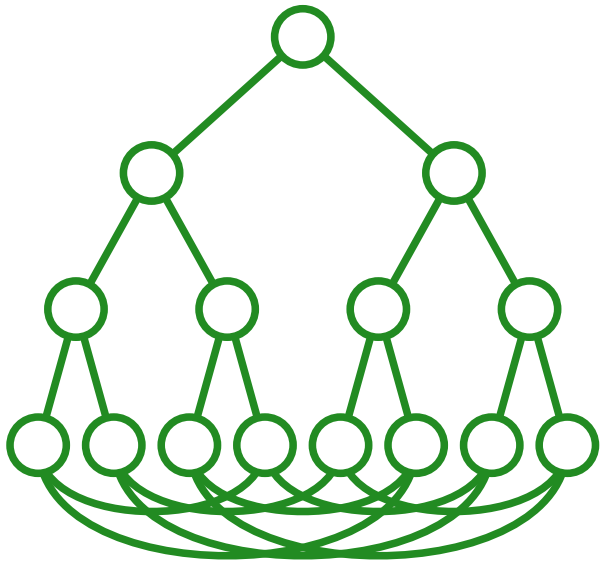

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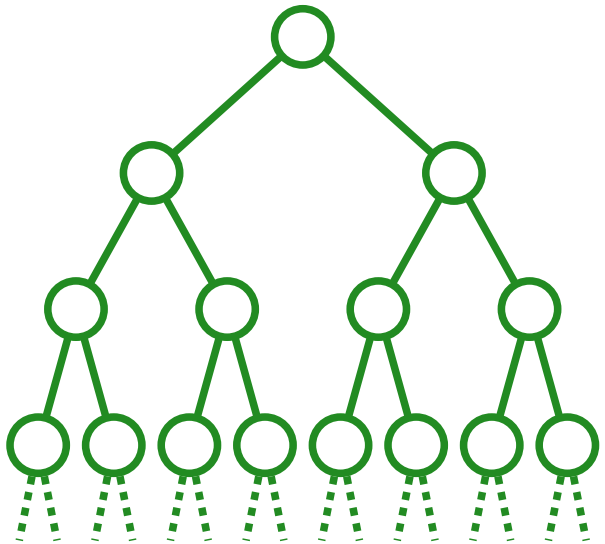

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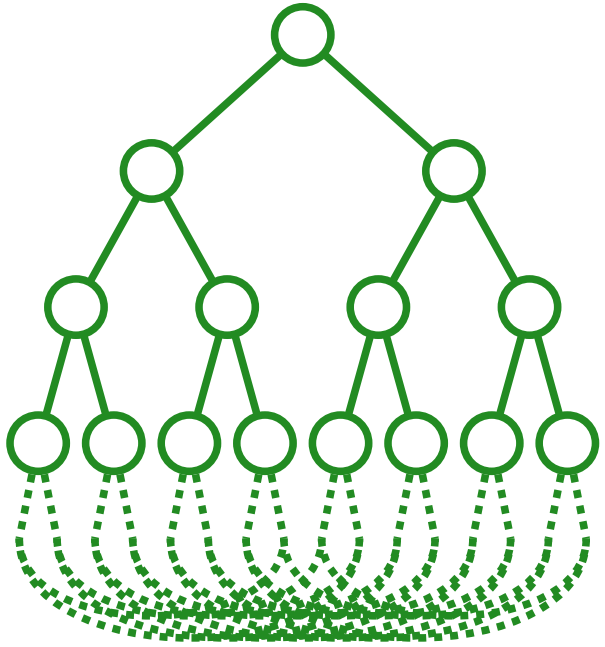

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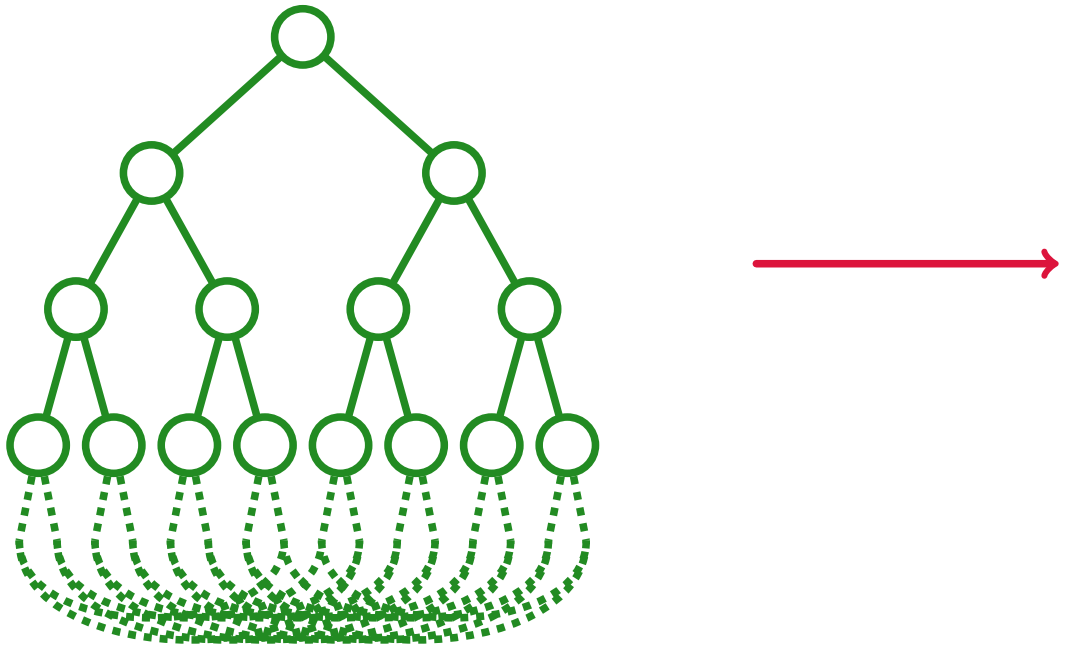

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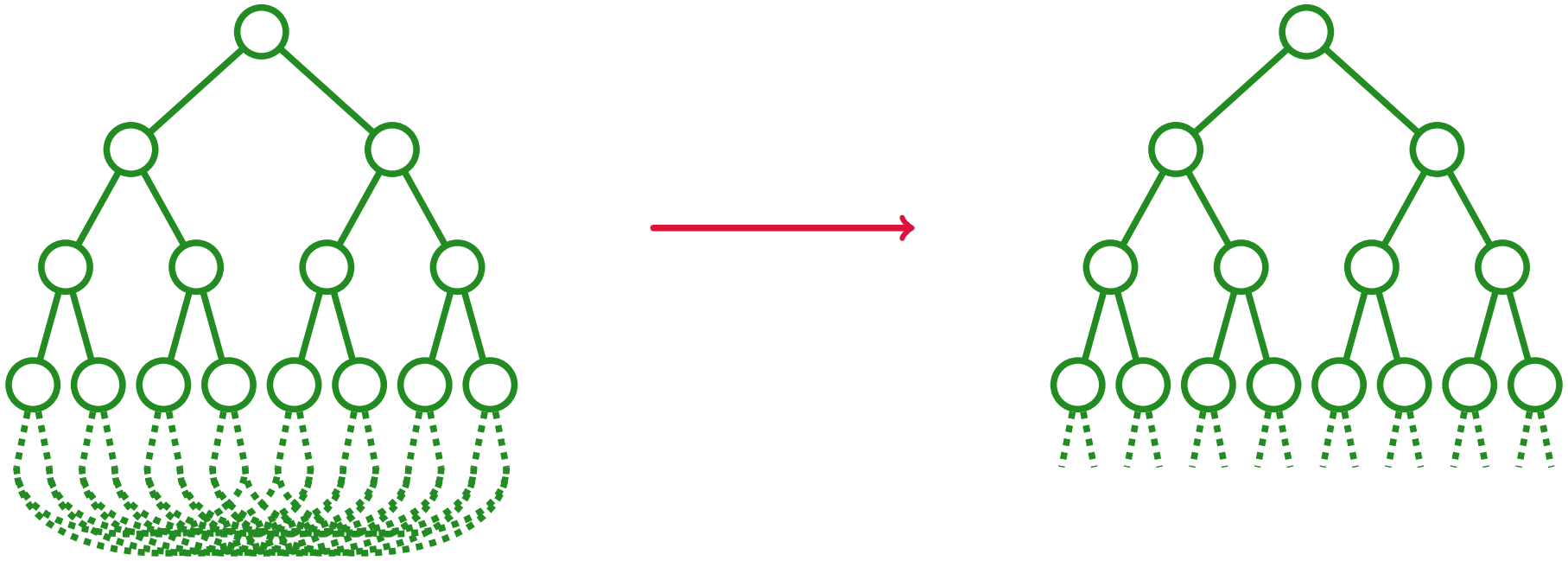

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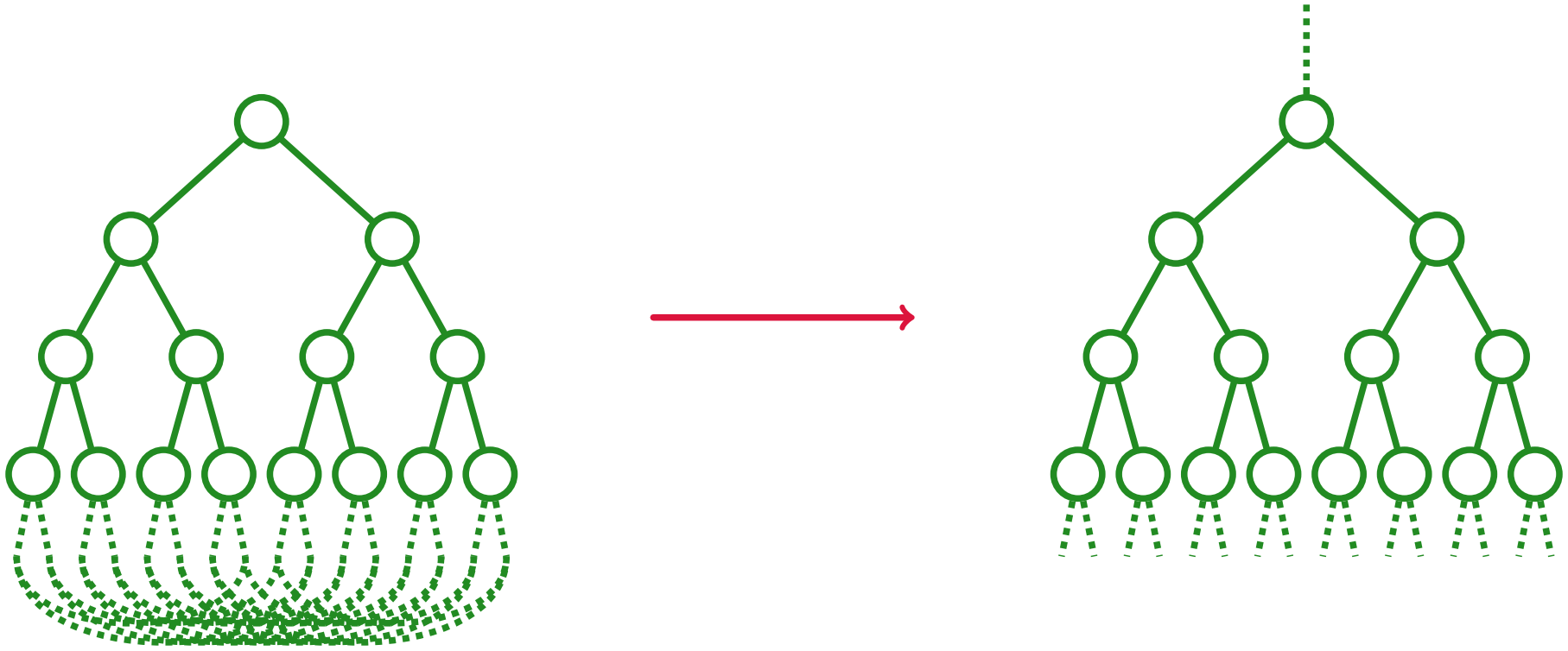

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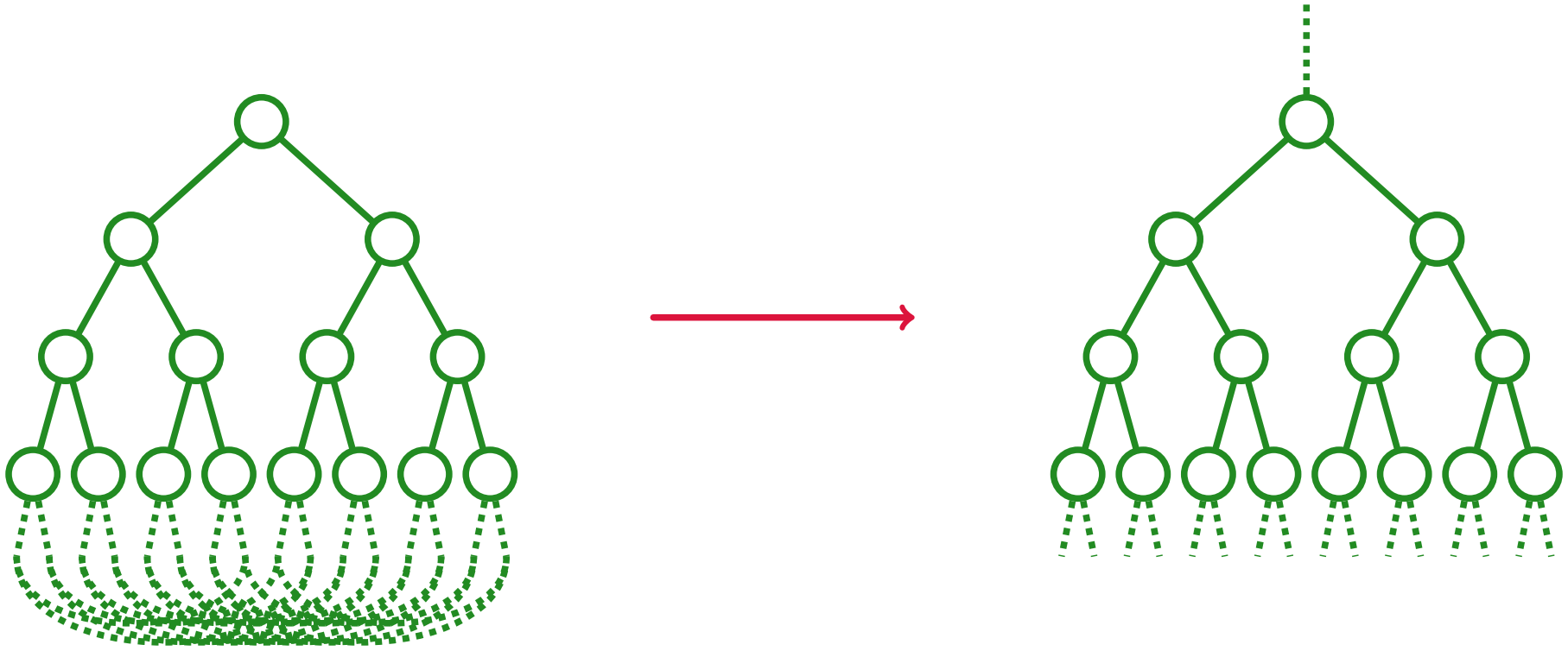
→ **Almost:** d-regular graphs with diverging girth converge to the infinite d-ary tree.



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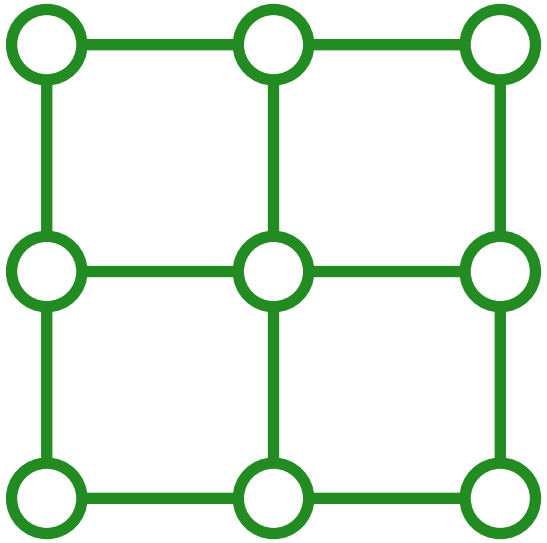

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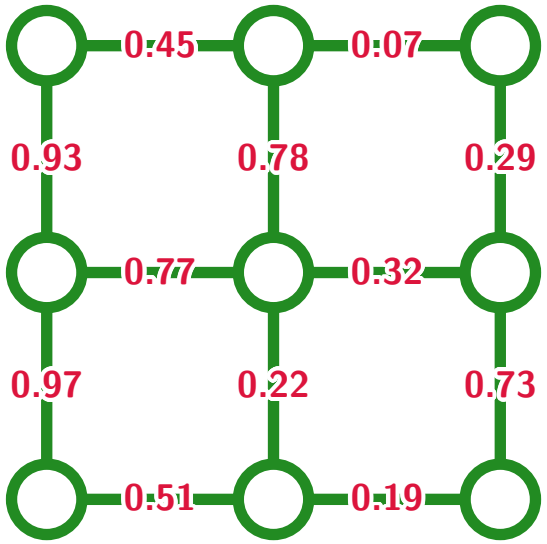

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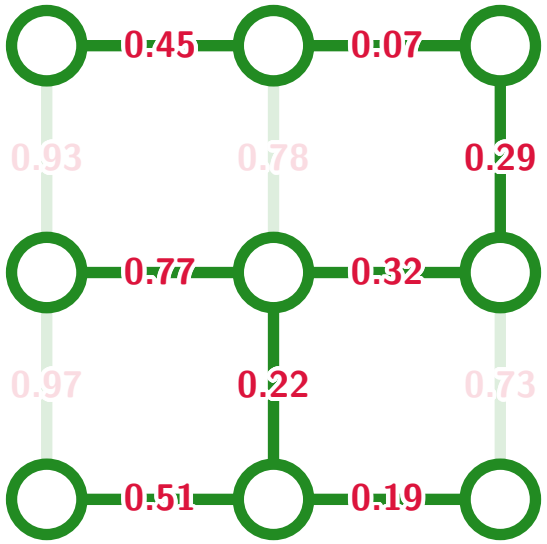

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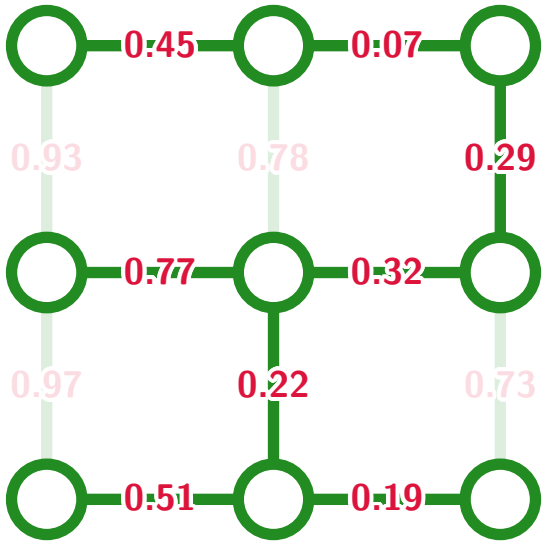
→ Can we find an "easy" way to compute the minimum spanning tree?



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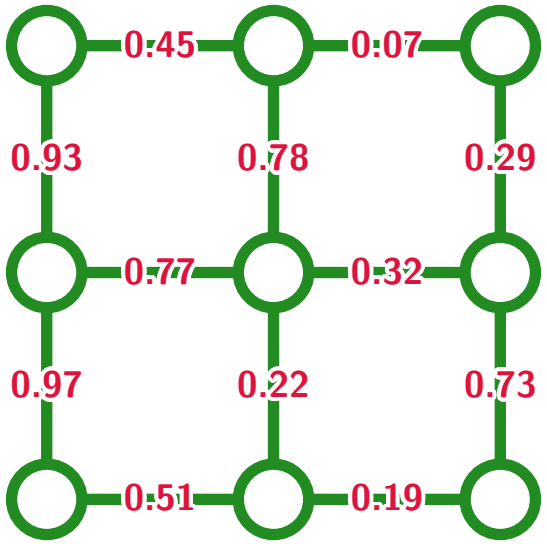

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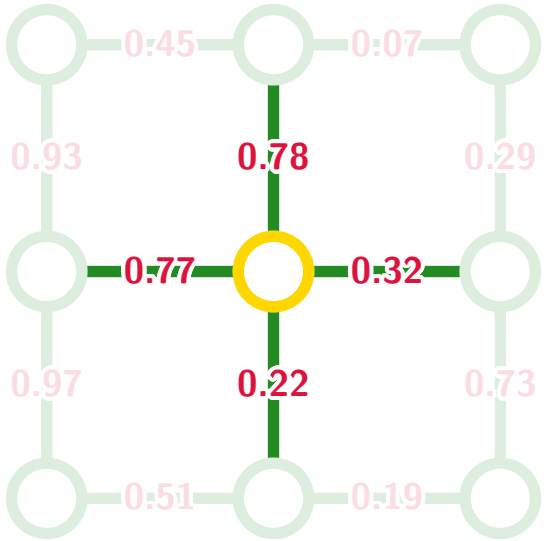

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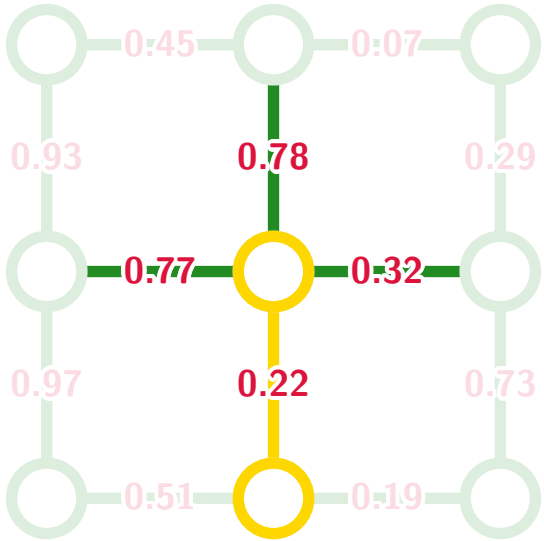

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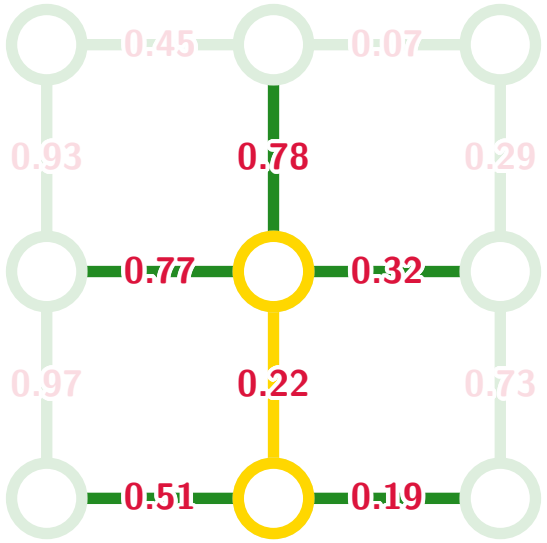

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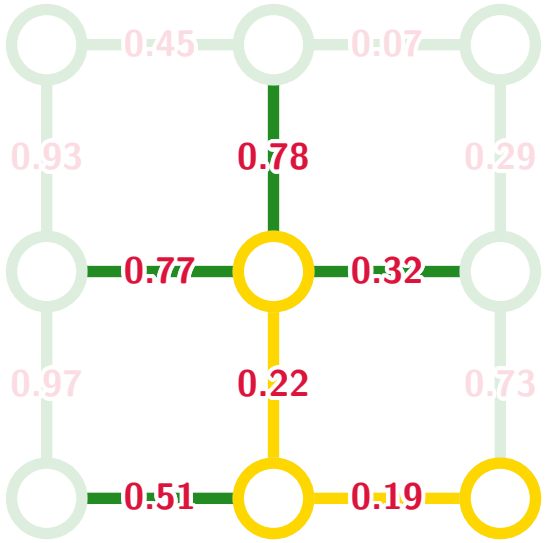

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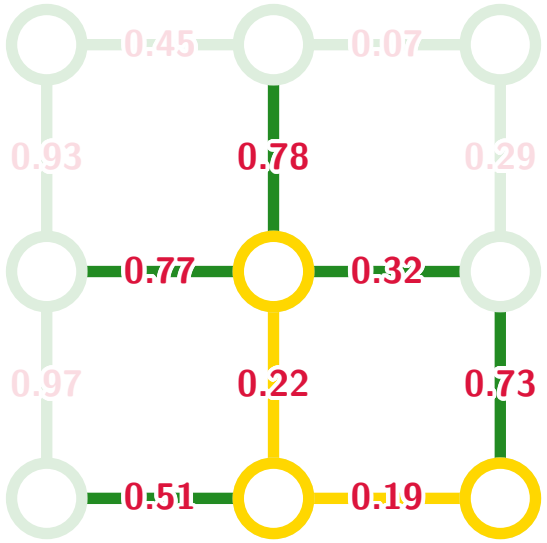

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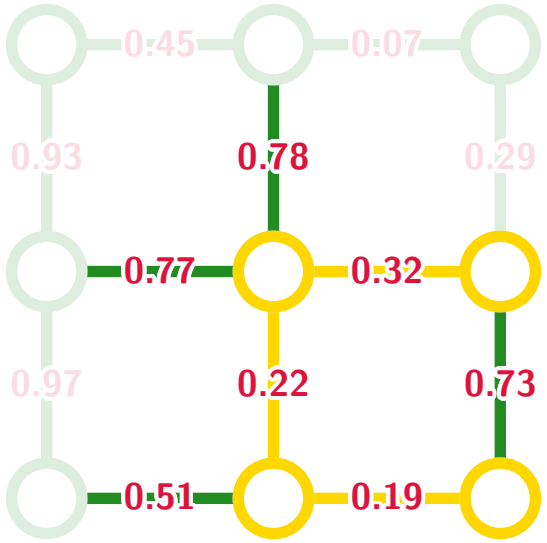

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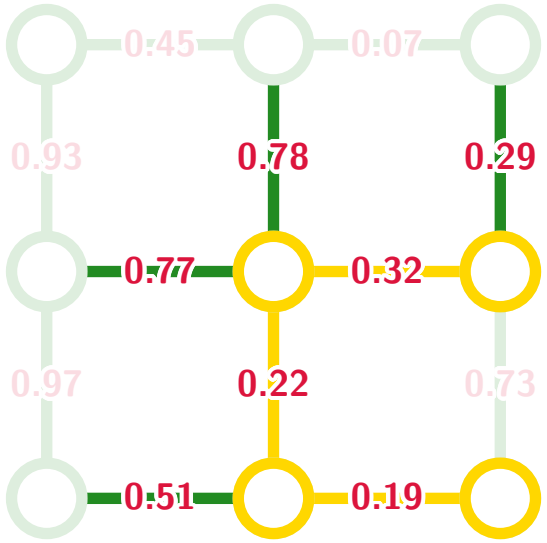

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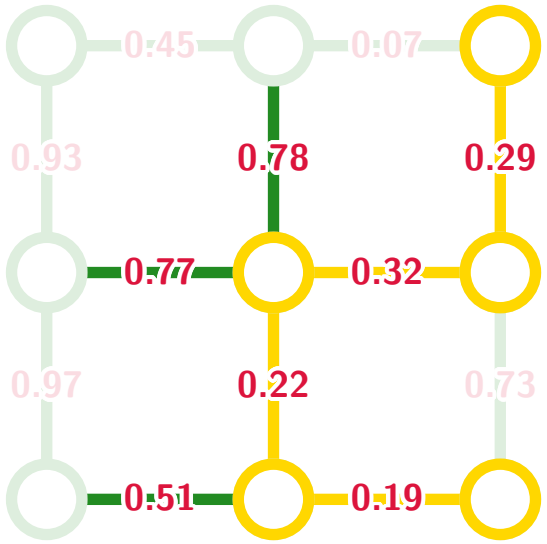

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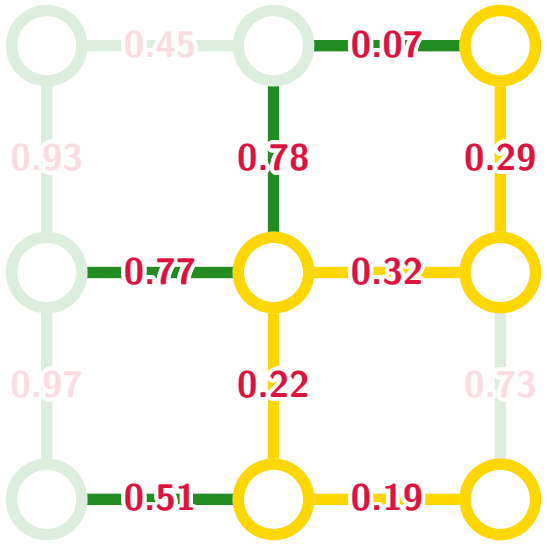

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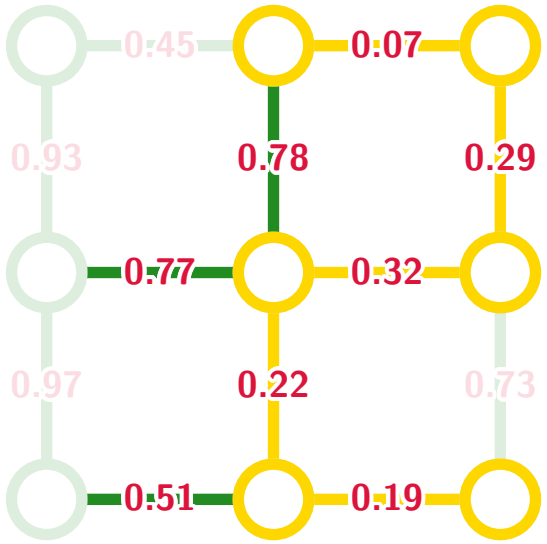

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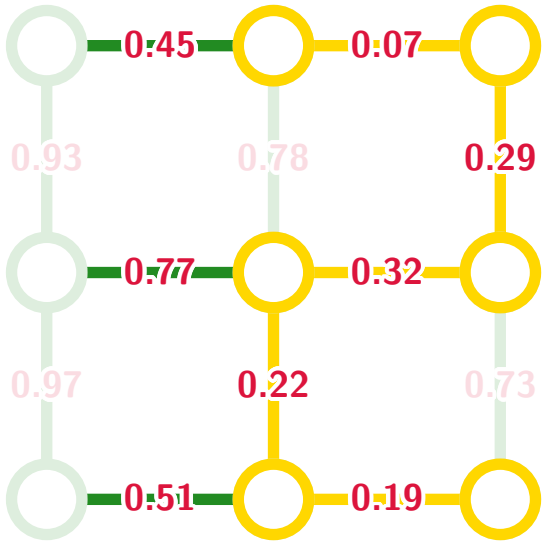

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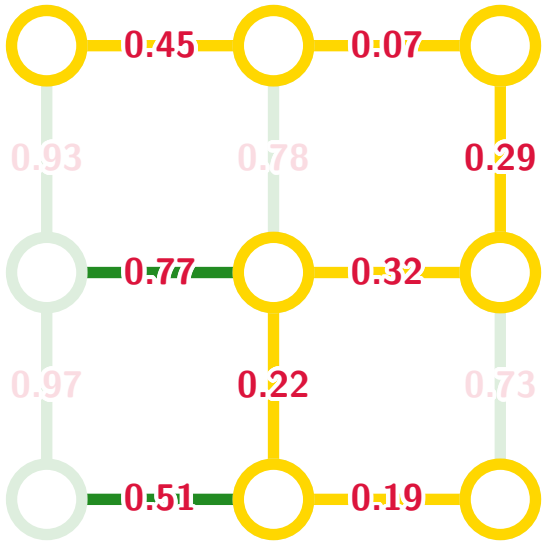

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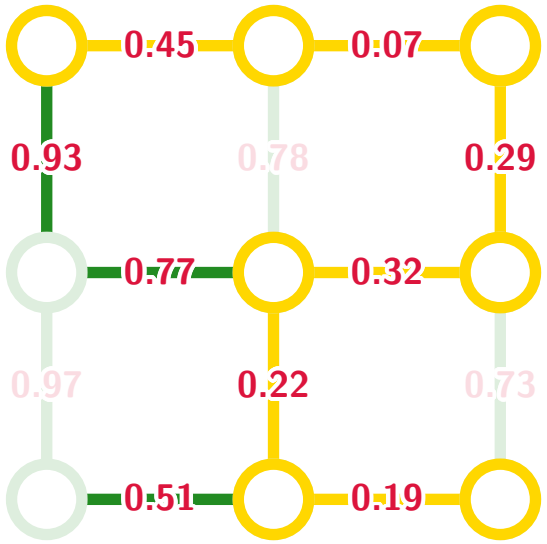

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- Start with a tree composed of a single node, arbitrarily chosen from the graph.
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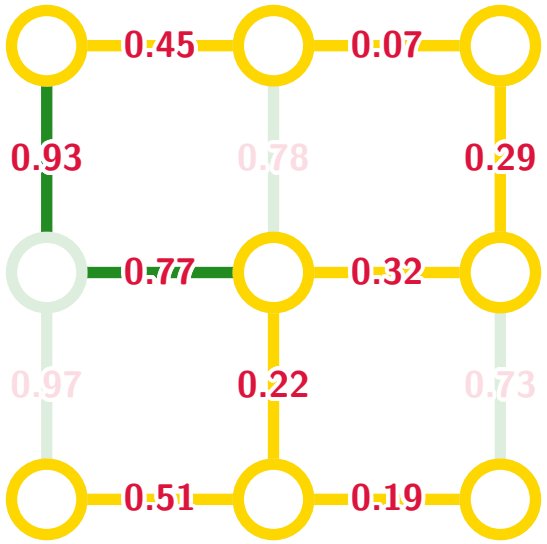

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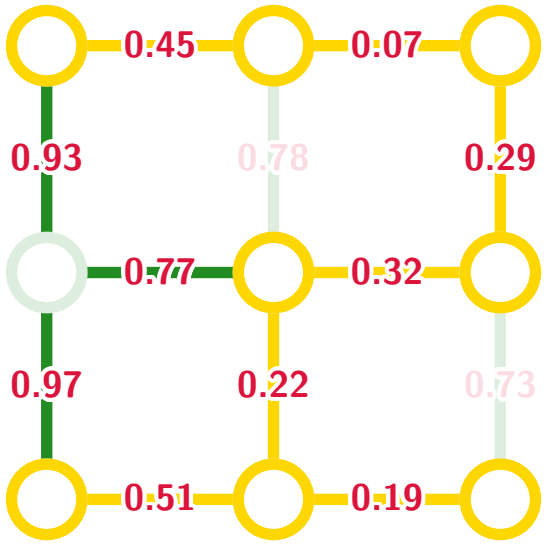

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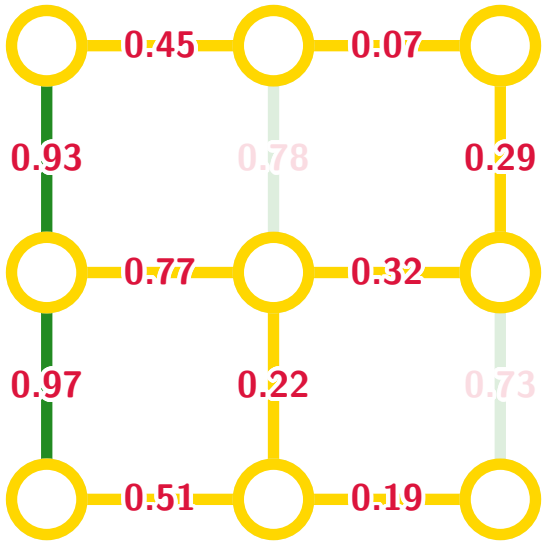

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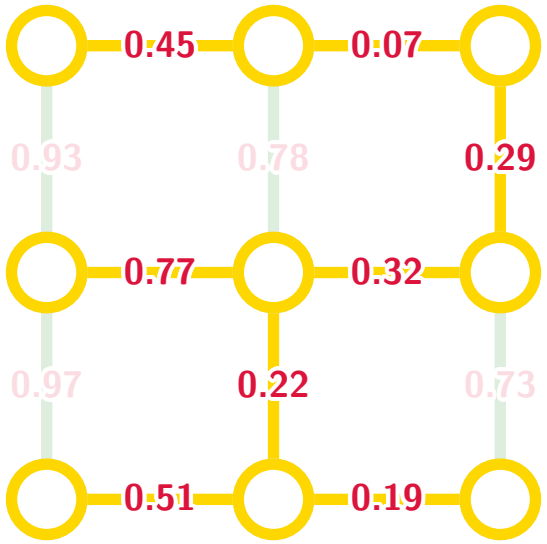

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Proposition

The edges of the minimum spanning tree are exactly those that are never the heaviest on any cycle.

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→ The minimum spanning tree is the entire line.

→ Prim's algorithm only explores the right portion of the line.

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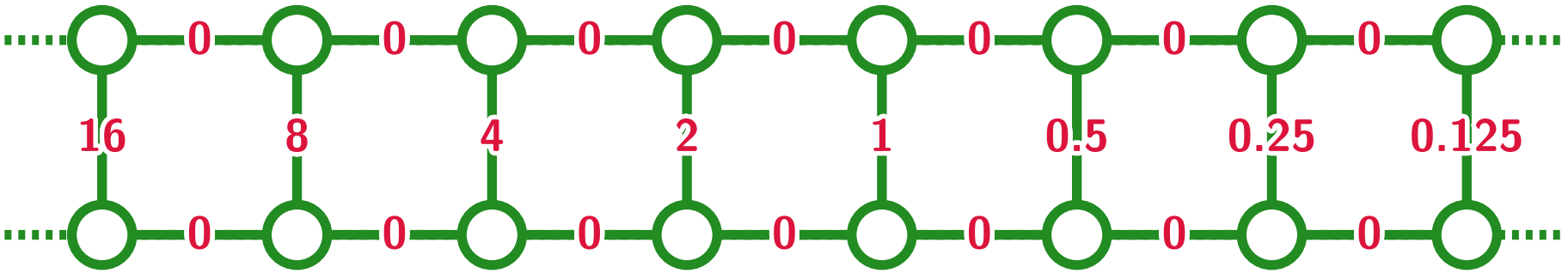

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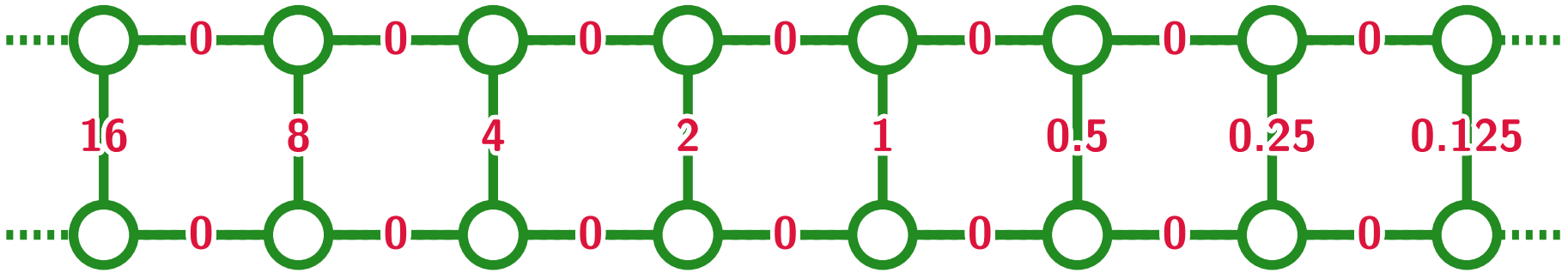

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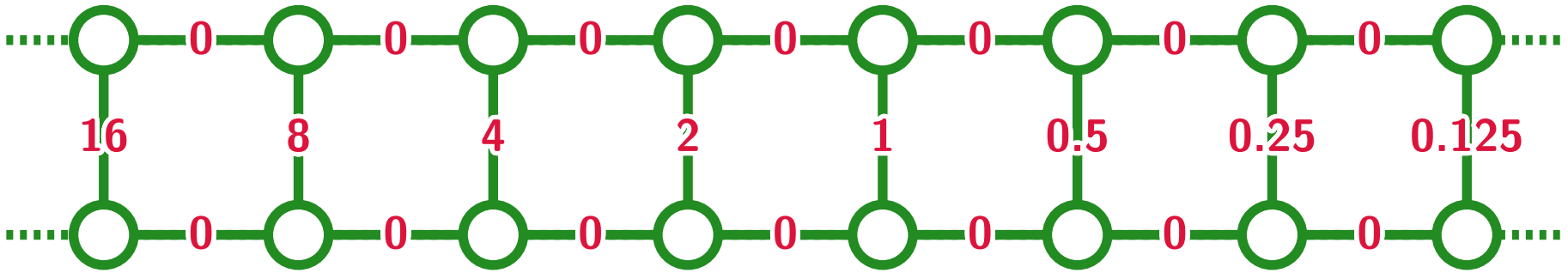

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- It is often referred to as the *invasion percolation cluster*.

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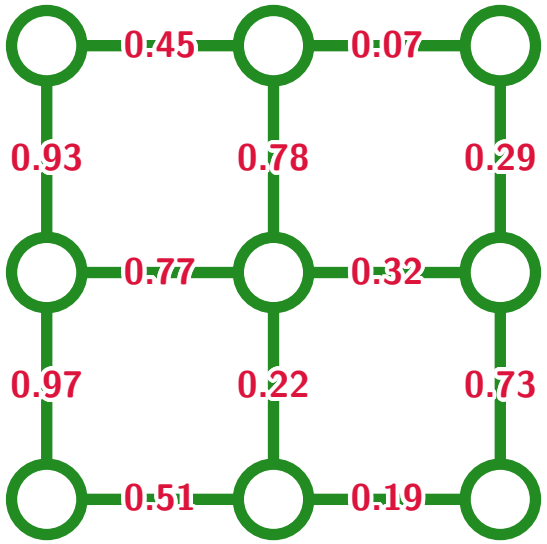

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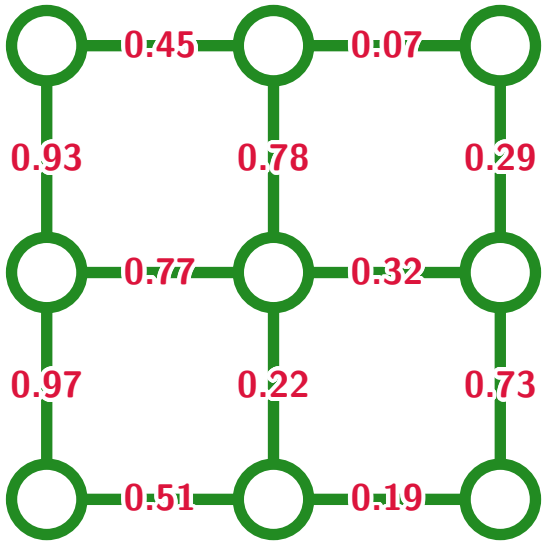

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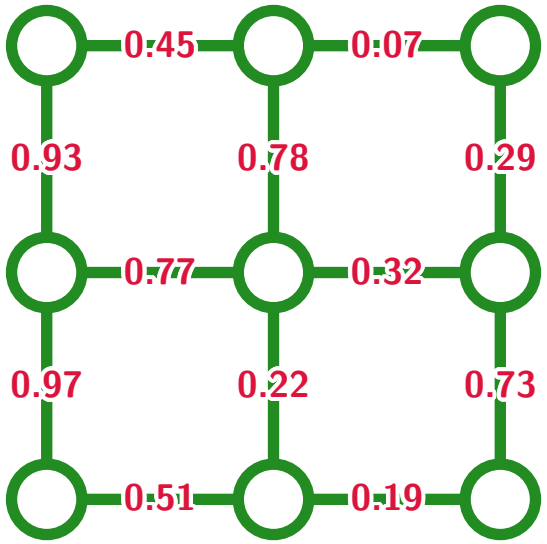

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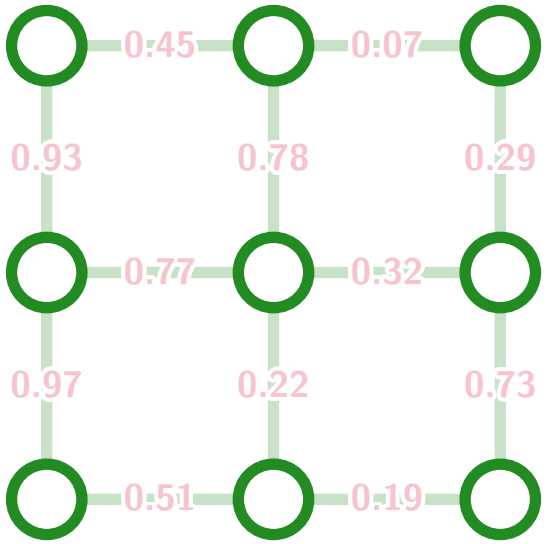

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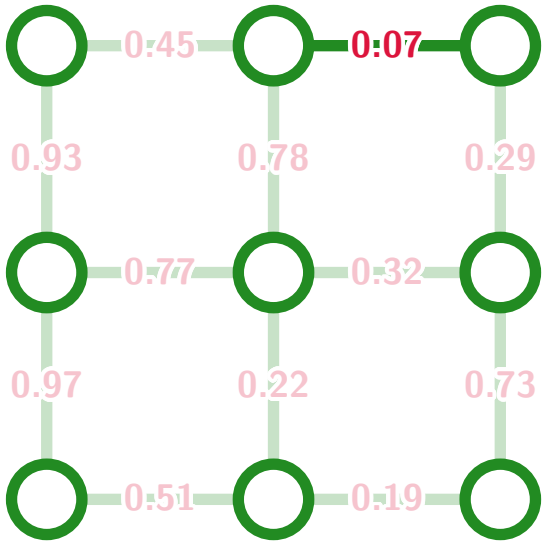

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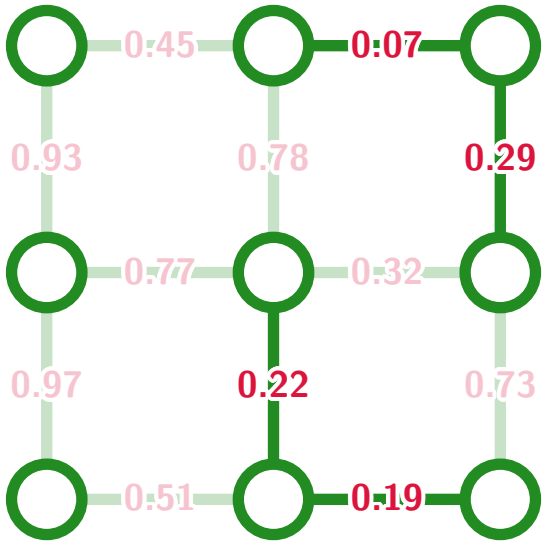

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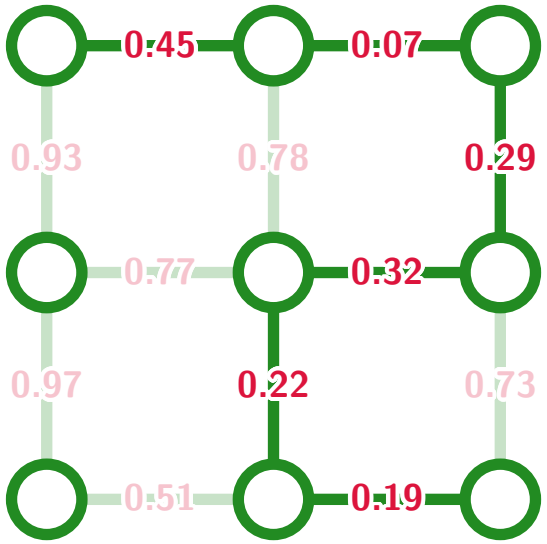

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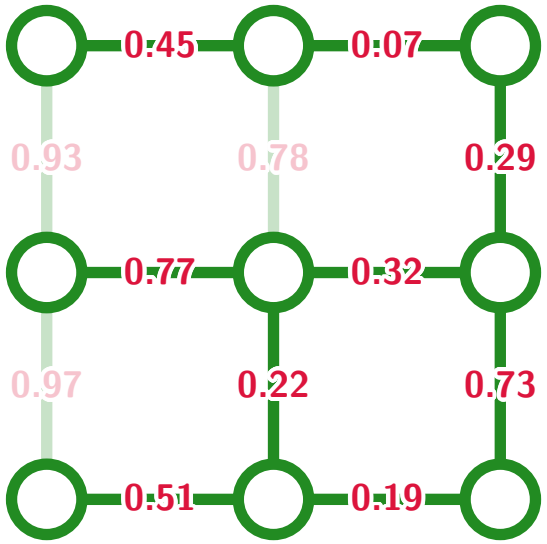

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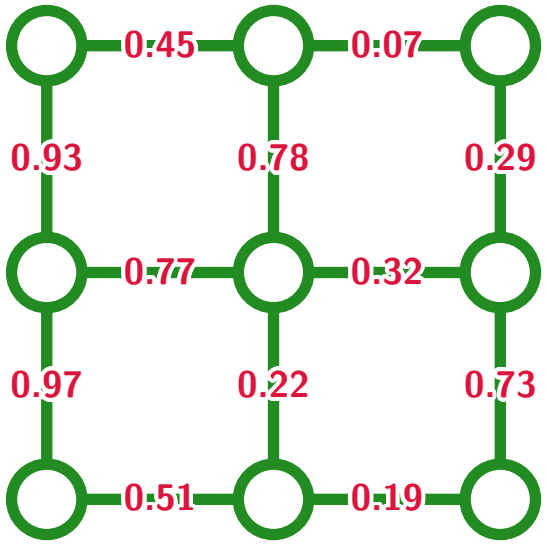

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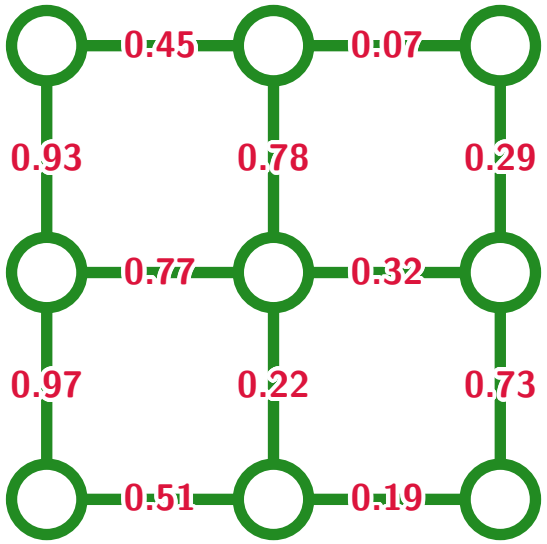

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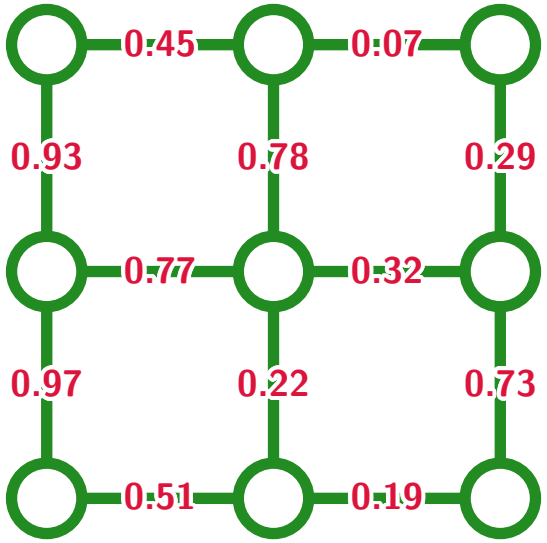

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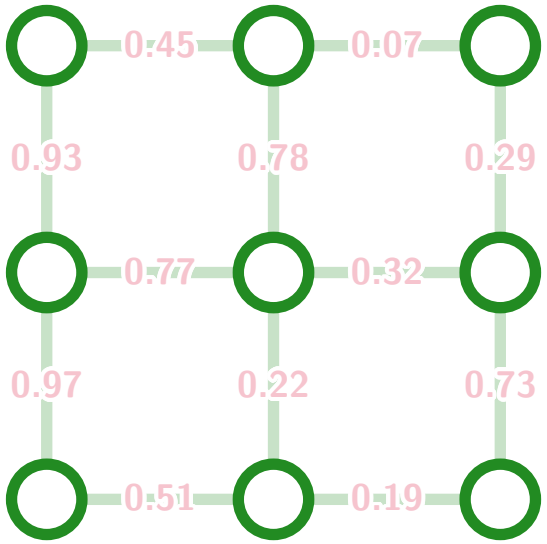

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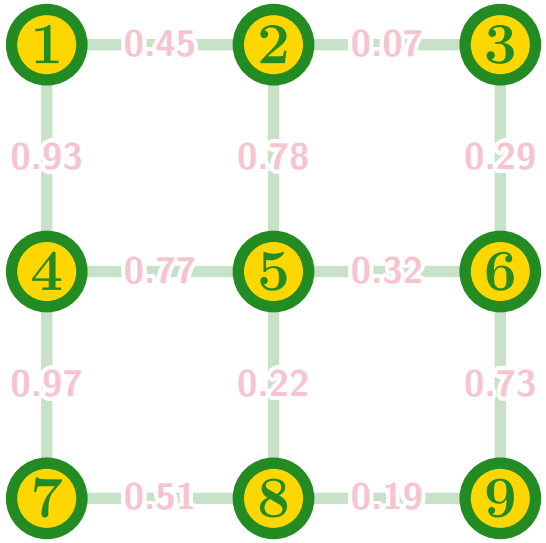

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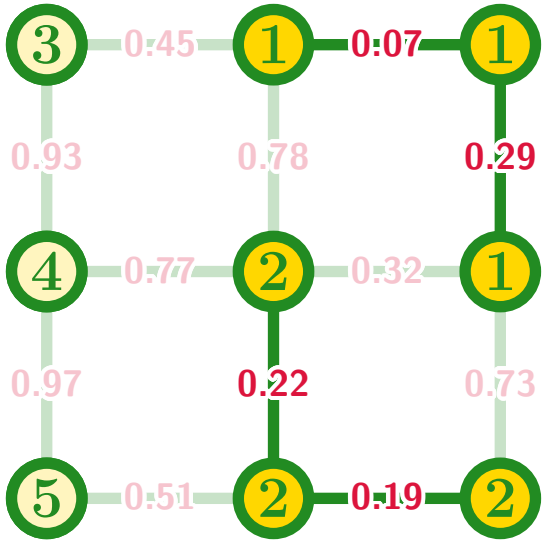

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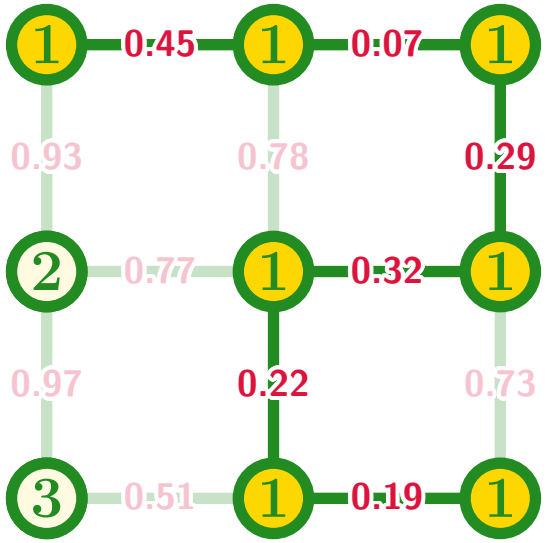

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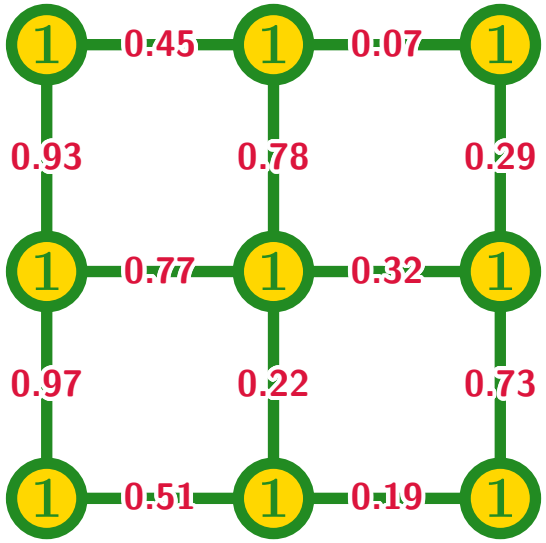

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A sequence of (finite) graphs $(G_n)_{n\geq 1}$ is *proper* if there exists $\theta : [0,1] \to [0,1]$ such that

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Furthermore, θ should be continuous on $(p_c, 1] = \theta^{-1}((0,1])$.

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→ Proper pairs can be seen as sequences of graphs and their limits for which the largest finite components exactly correspond to an infinite component in the limit.

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- Most "natural" pairs of graphs and their limits are proper:

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→ Attach a 3- and a 4-regular graph together to obtain an improper pair, both finite and infinite.

Table of Content

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Let $(G_n)_{n\geq 1}$ and G be a proper pair. Then, the output of Prim's algorithm on $(G_n)_{n\geq 1}$ after $tn + o(n)$ steps for any $t \in [0, 1]$ converges towards the union of the invasion percolation cluster on G along with the minimum spanning forest on G percolated at level $\theta^{-1}(t)$.

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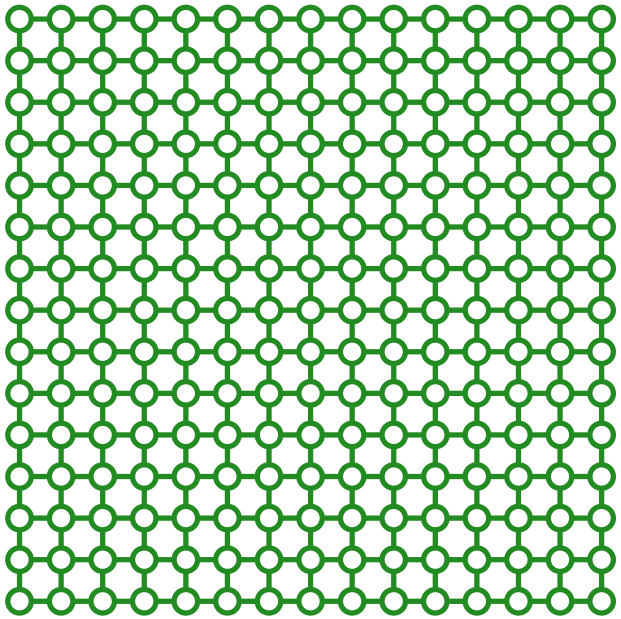
$\mathrm{Prim}\big(\theta(p)n\big) \longrightarrow \mathrm{IPC} + \mathrm{MST}(p)$

- Take a graph



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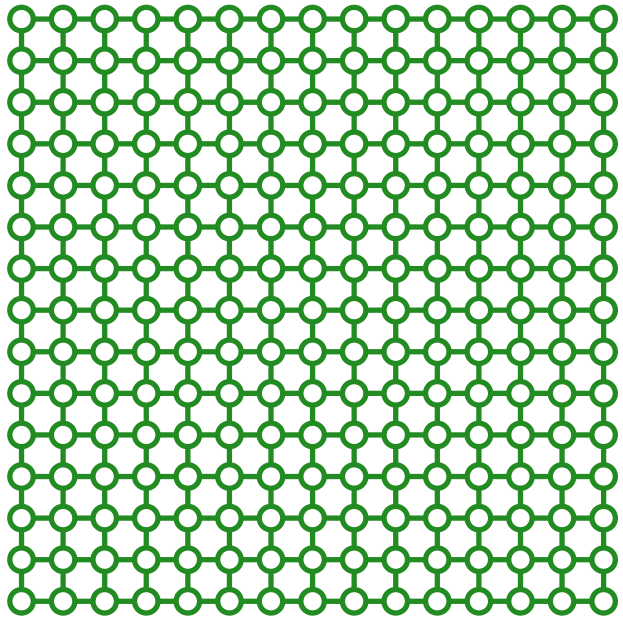

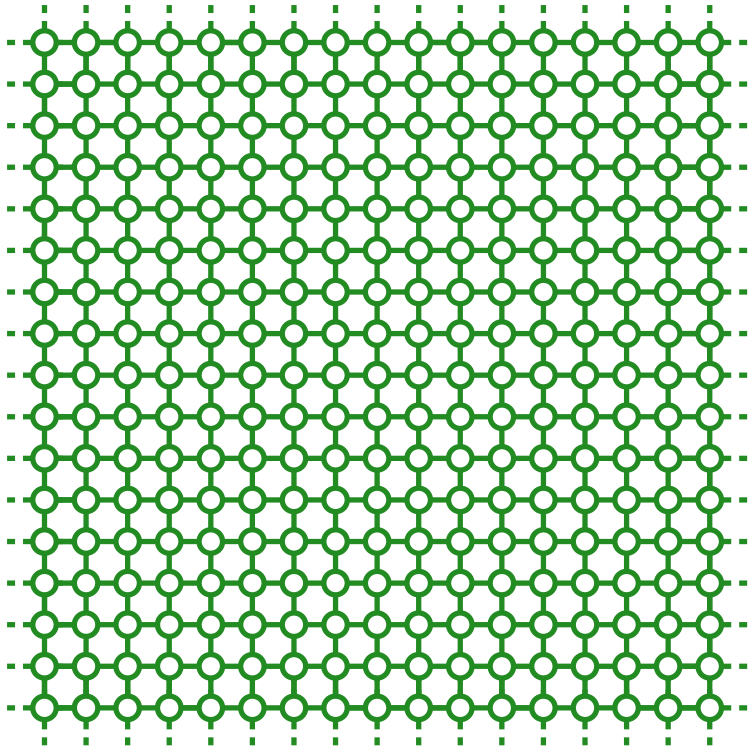

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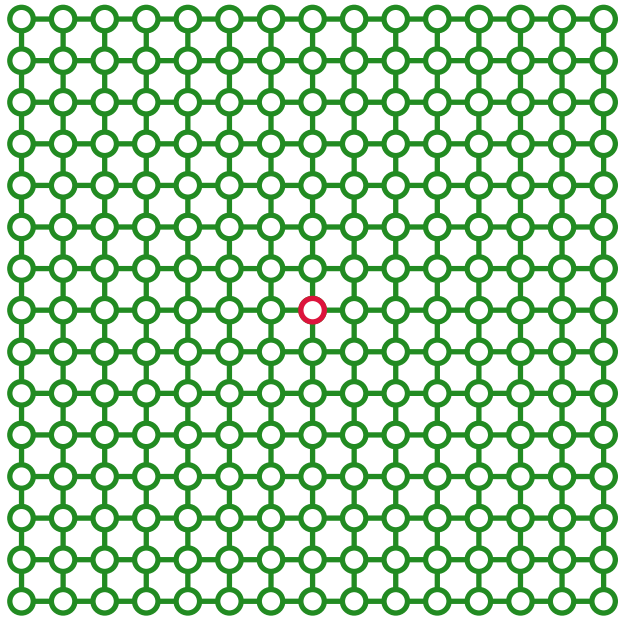

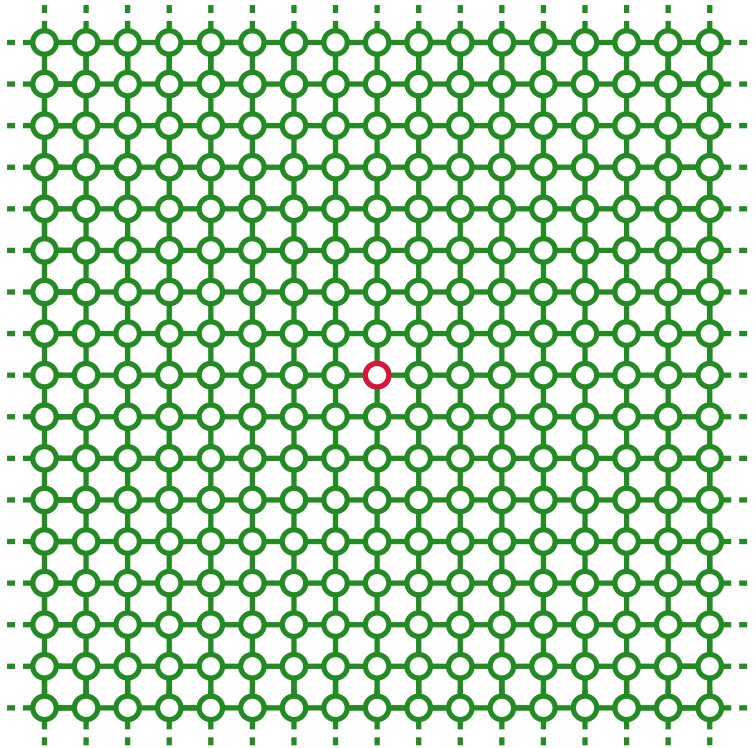

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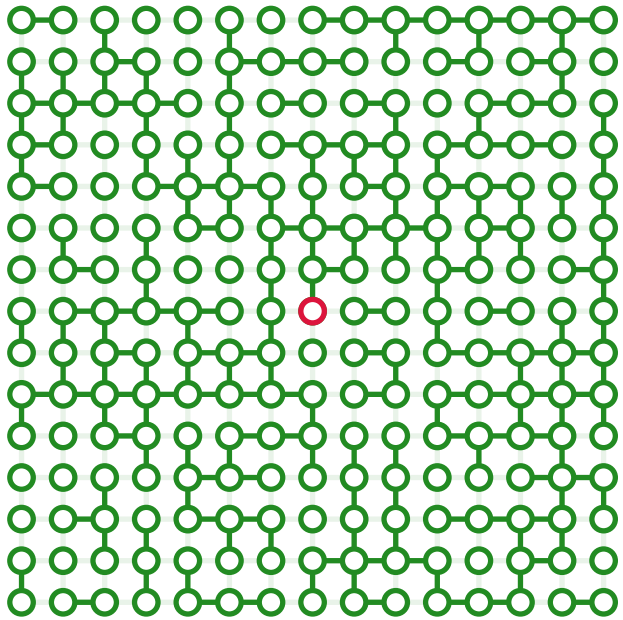

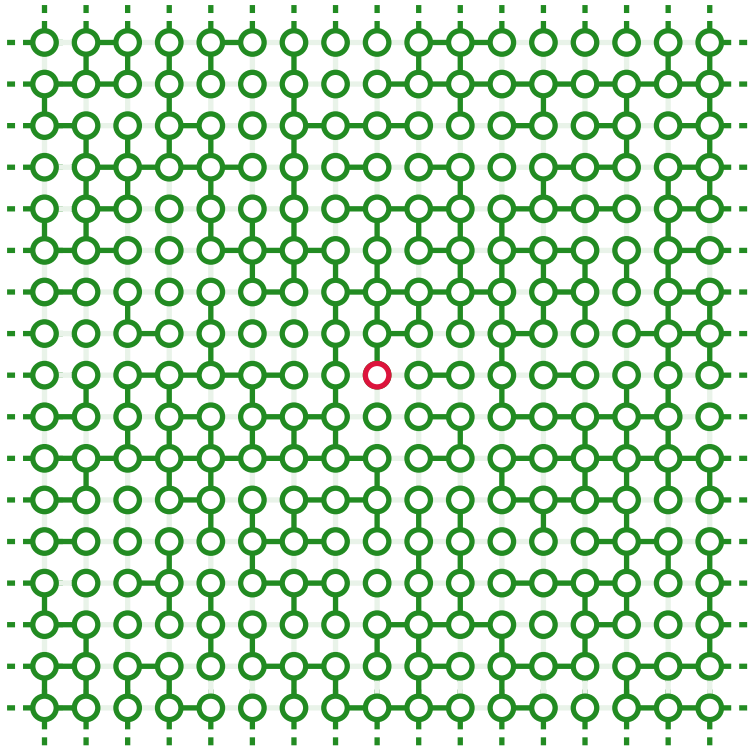

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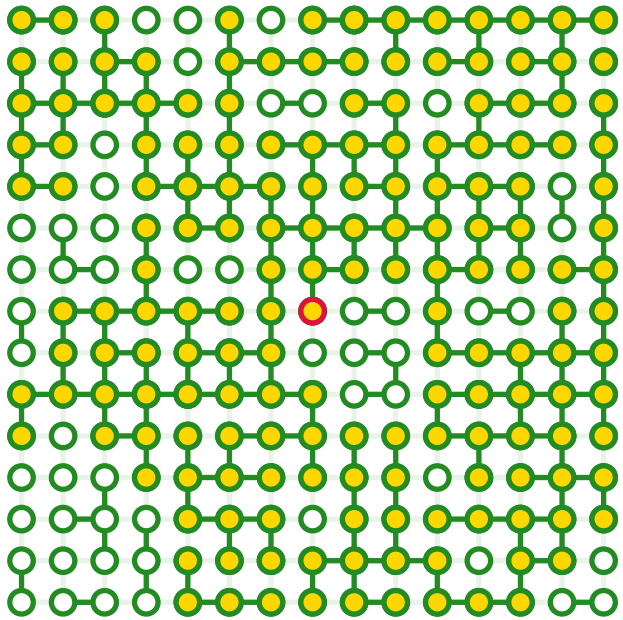

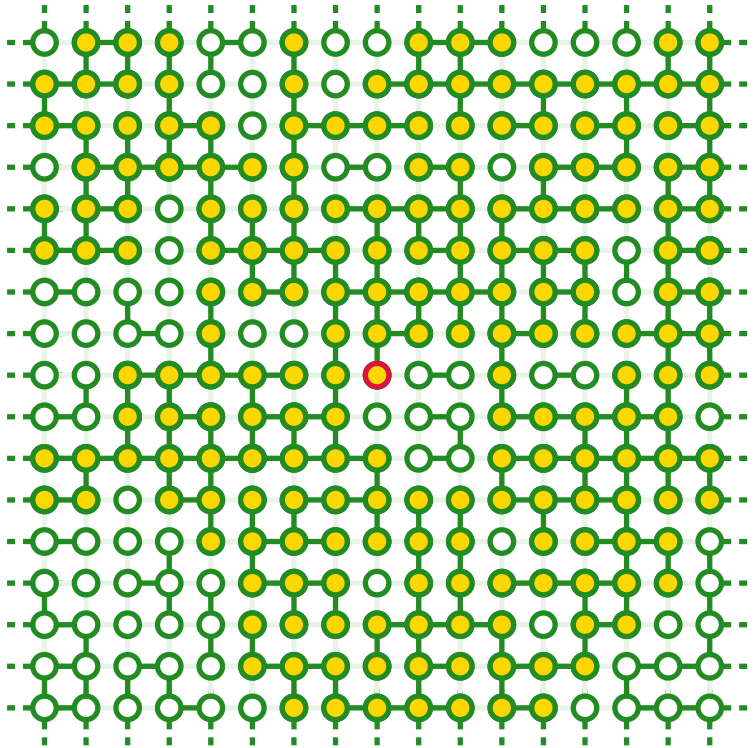

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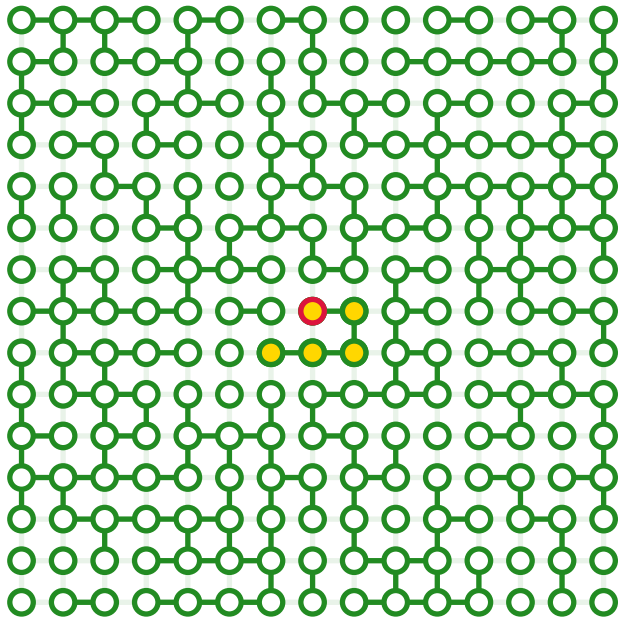

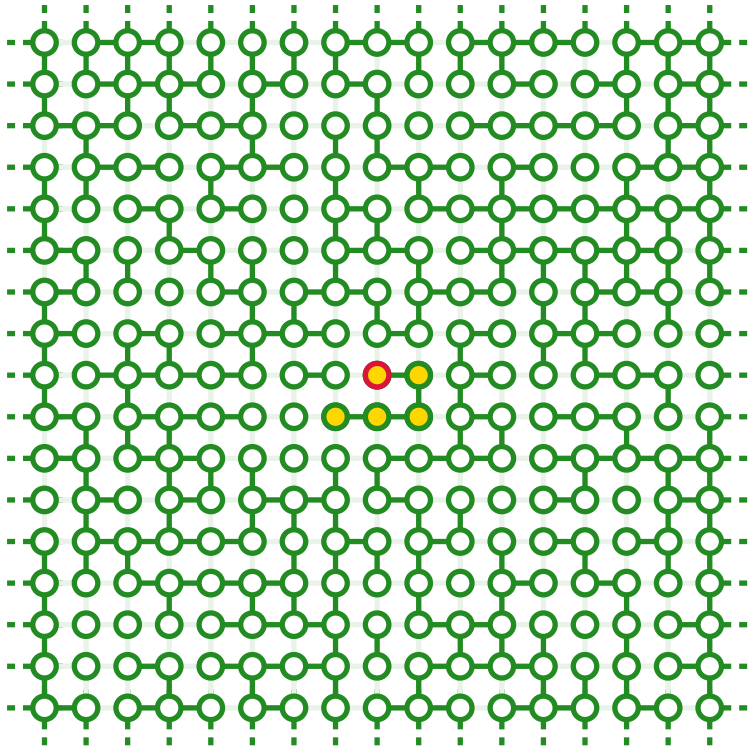

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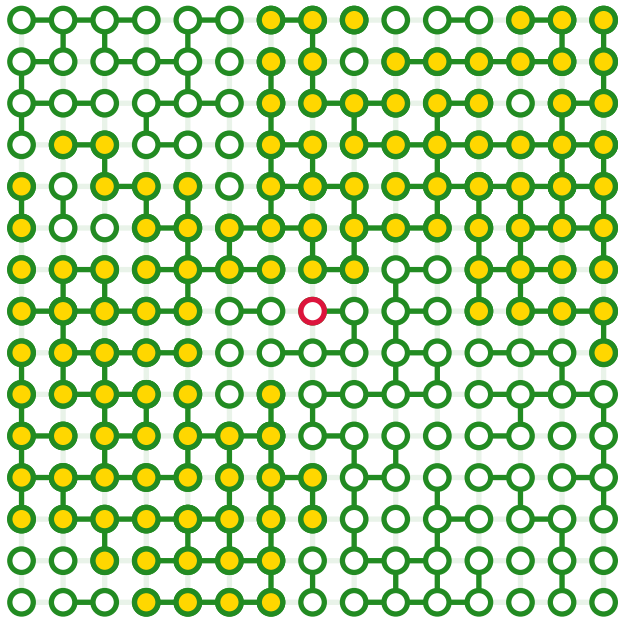

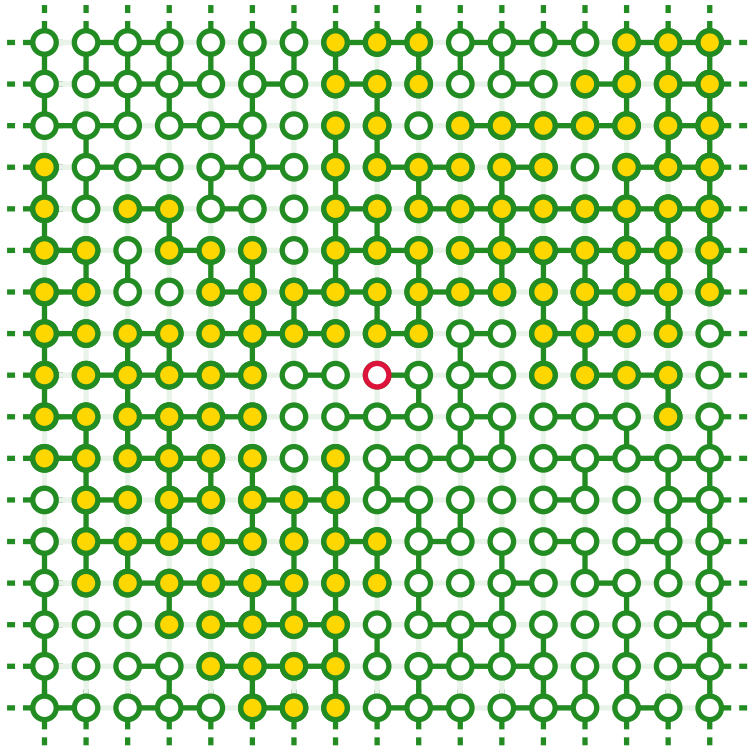

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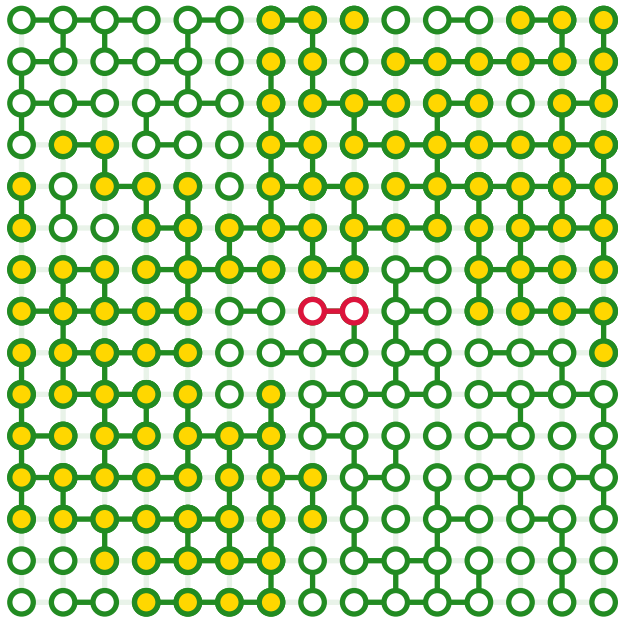

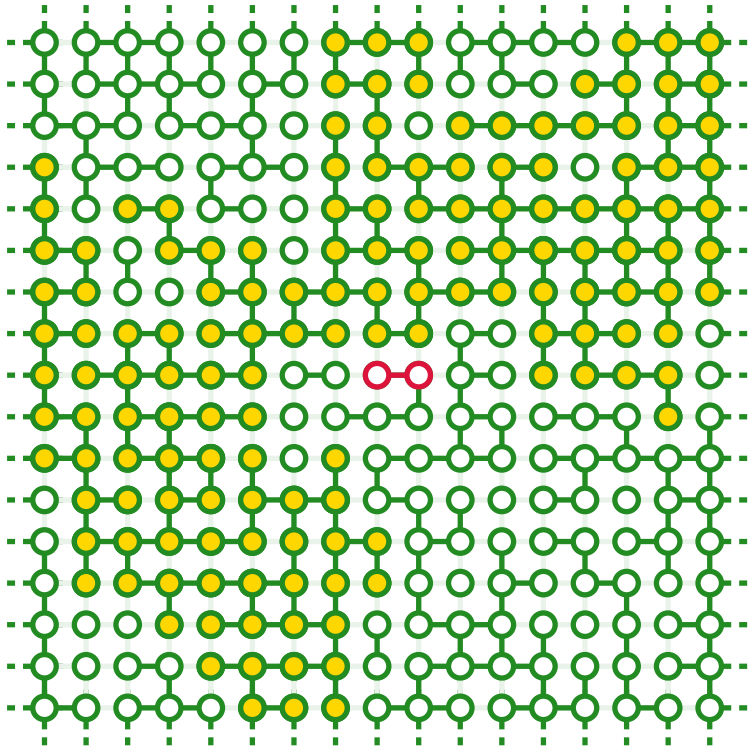

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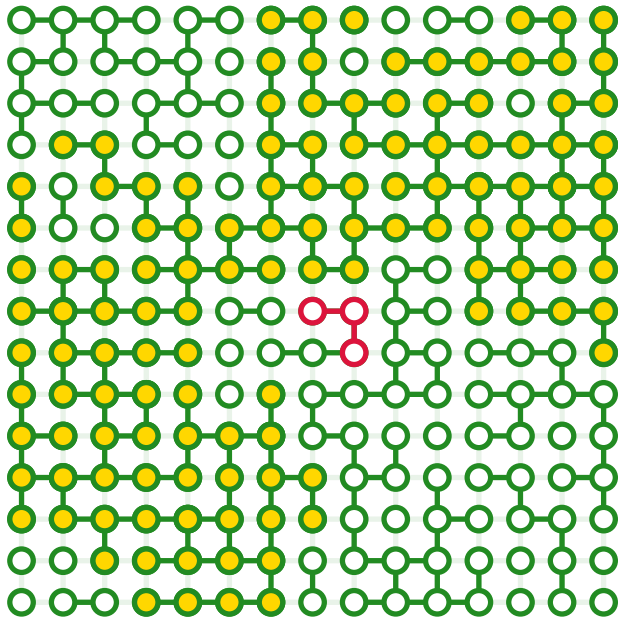

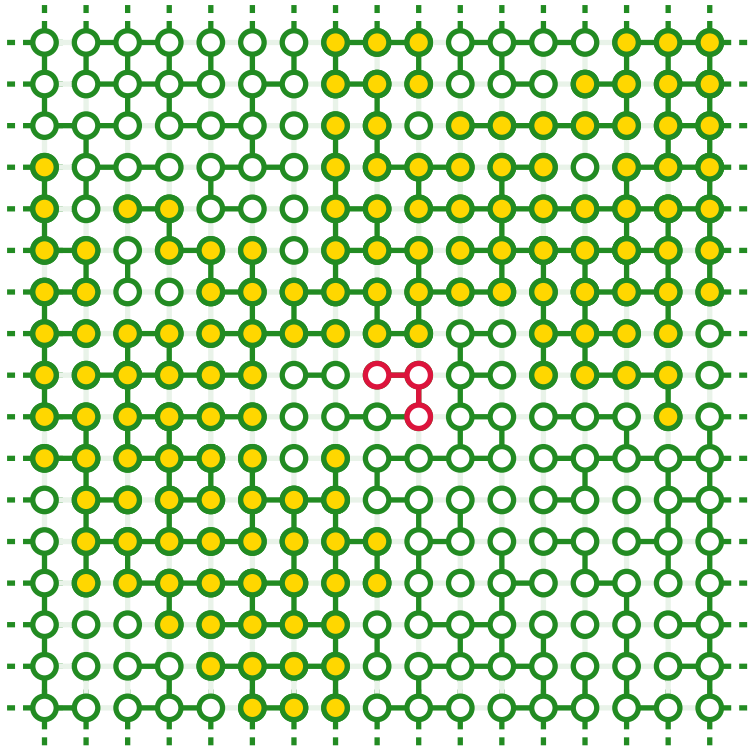

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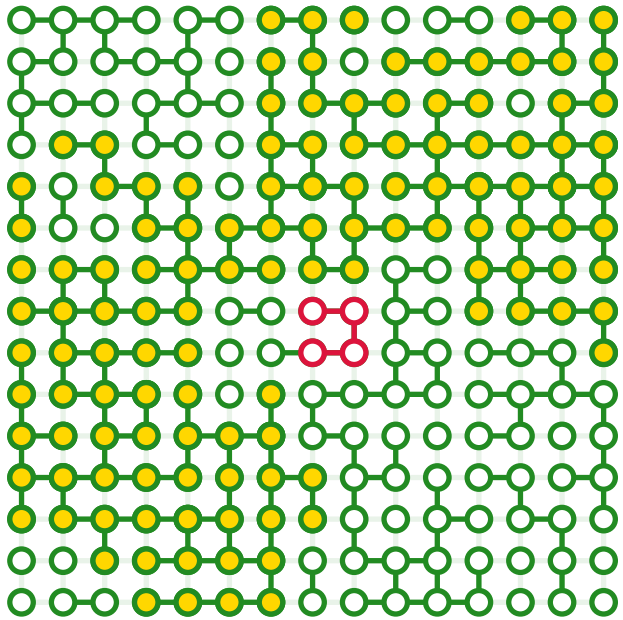

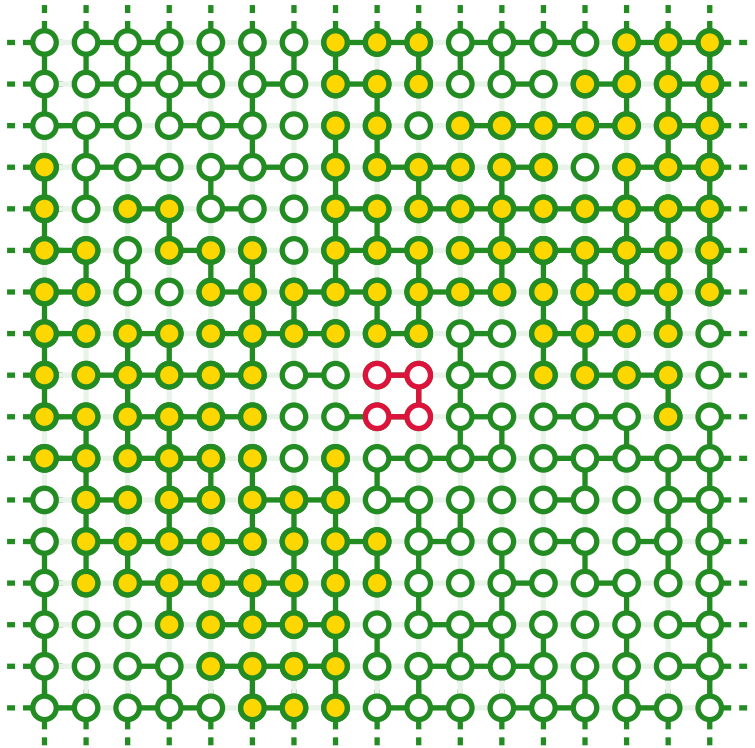

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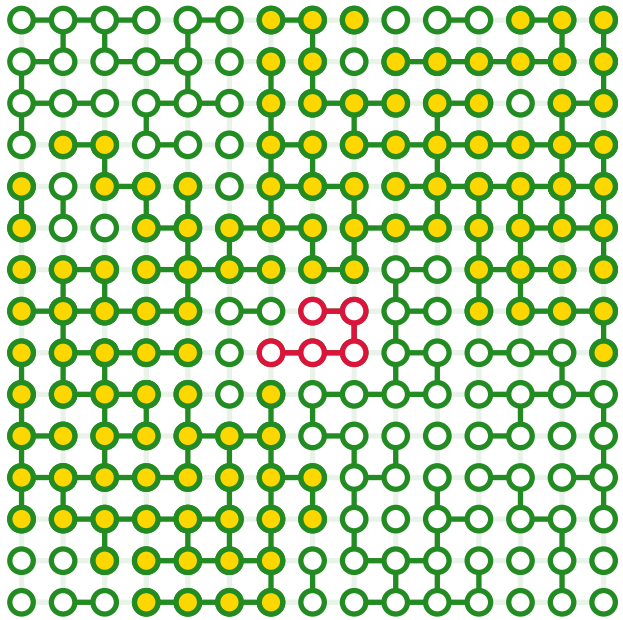

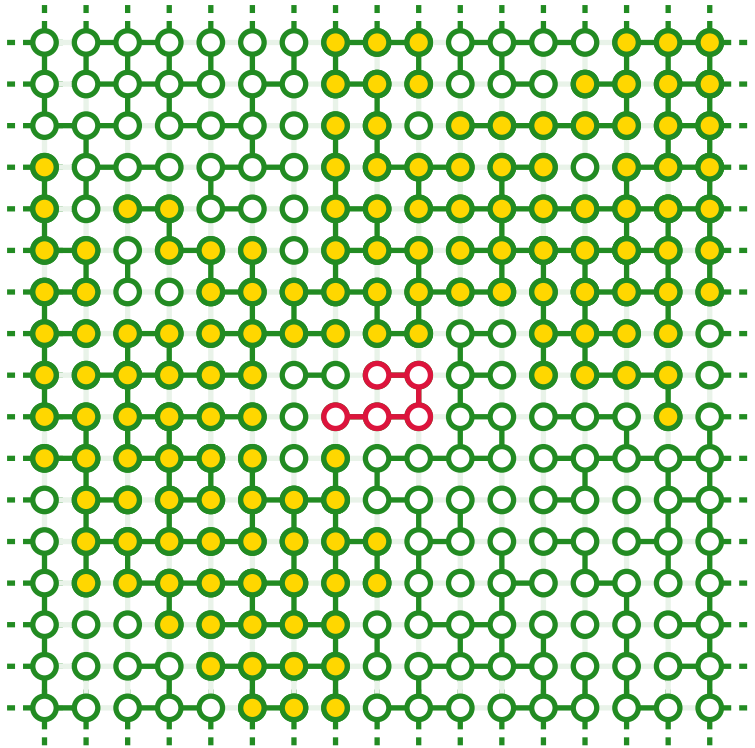

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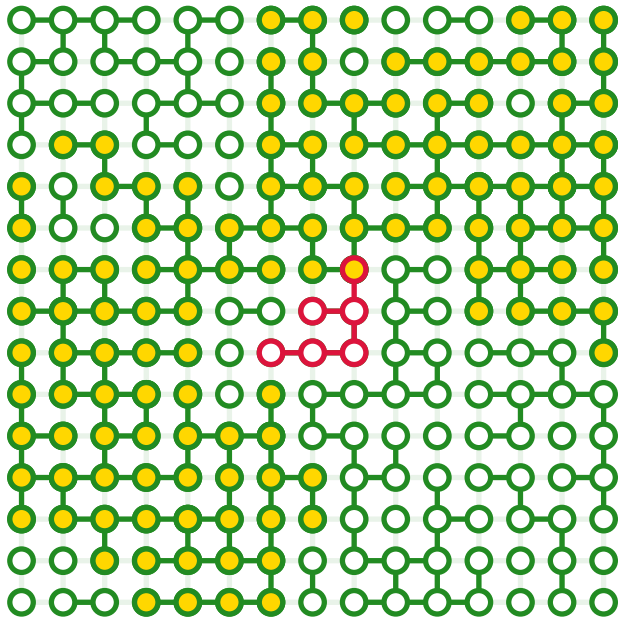

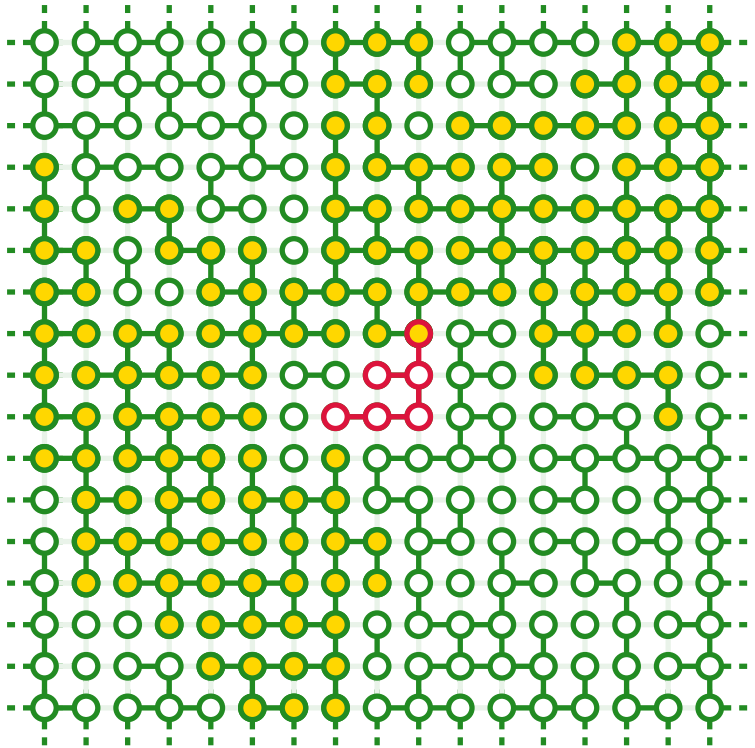

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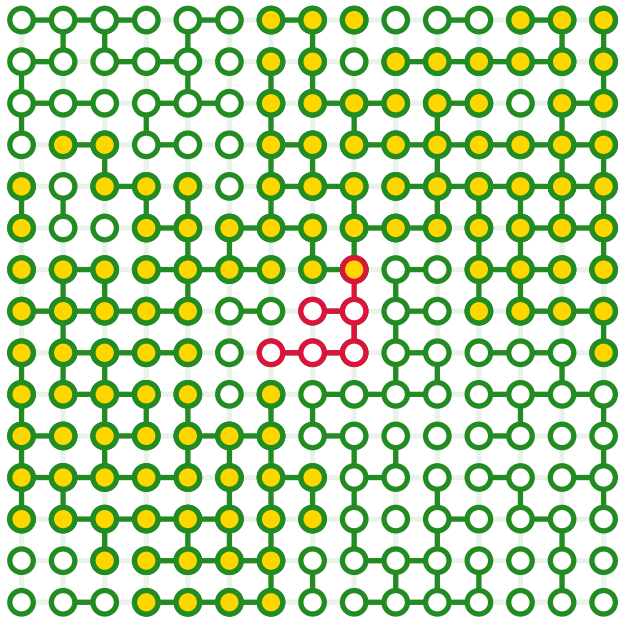

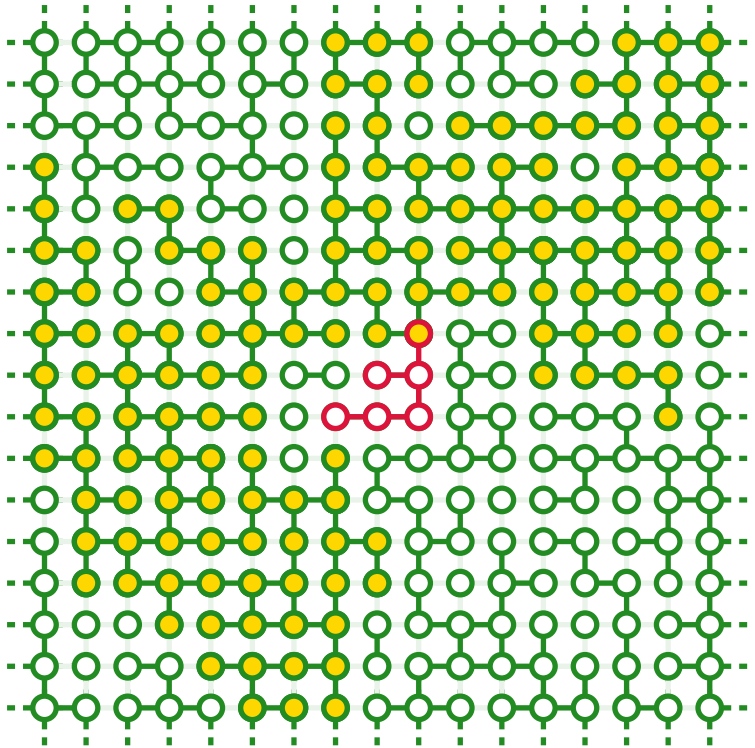

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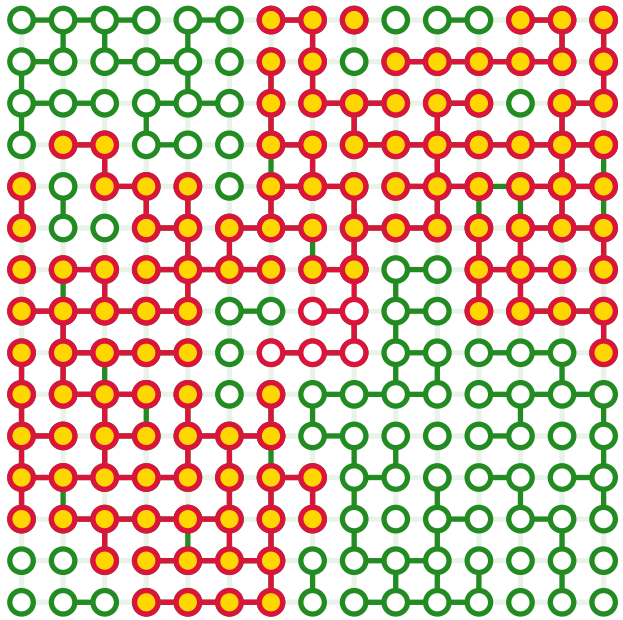

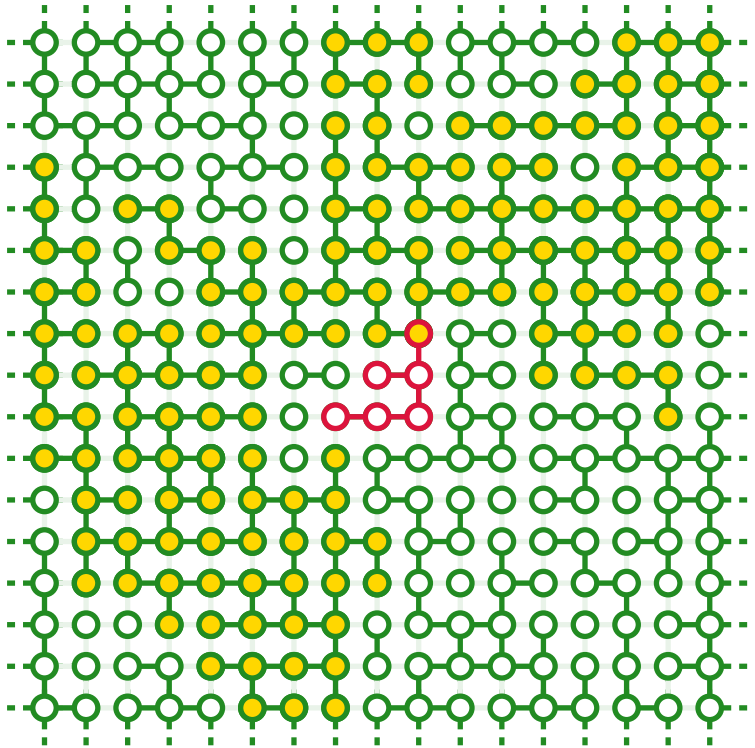

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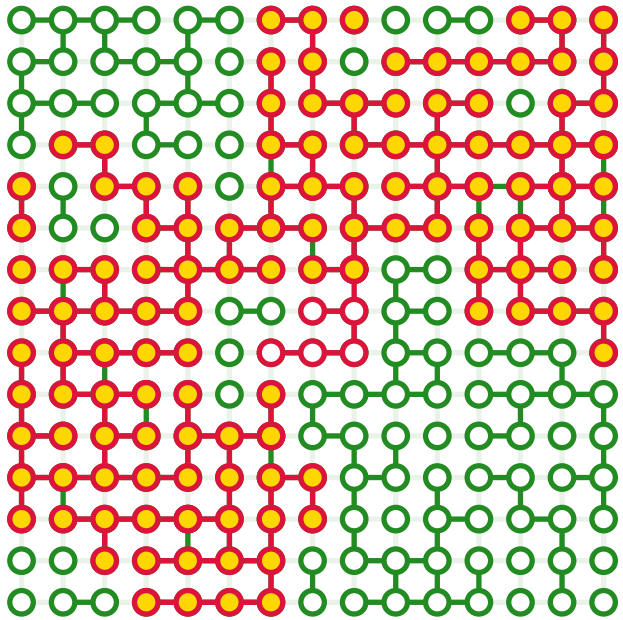

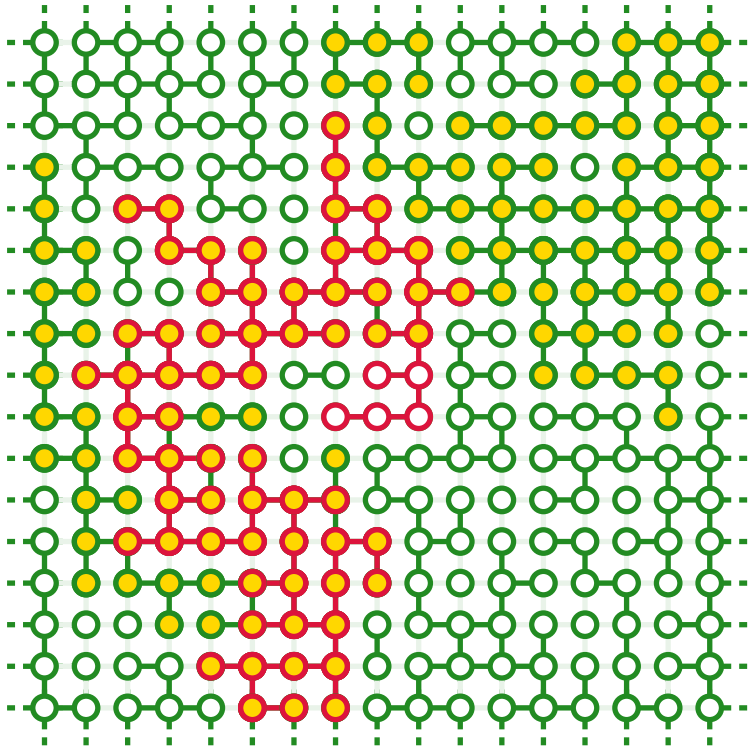

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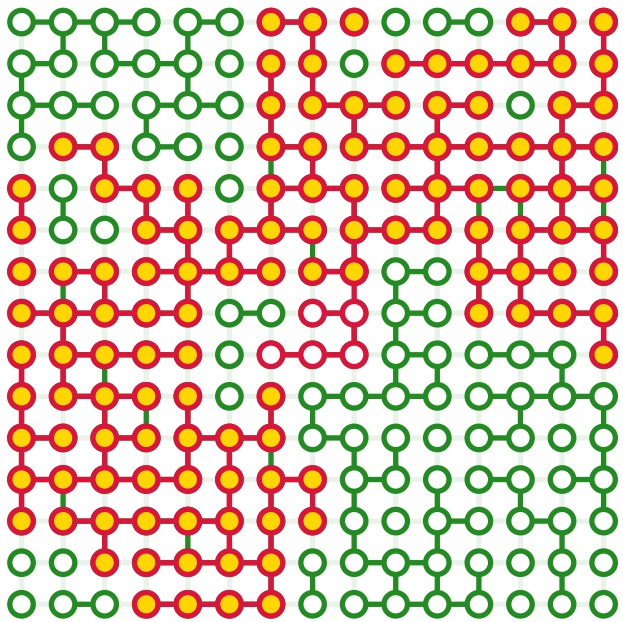

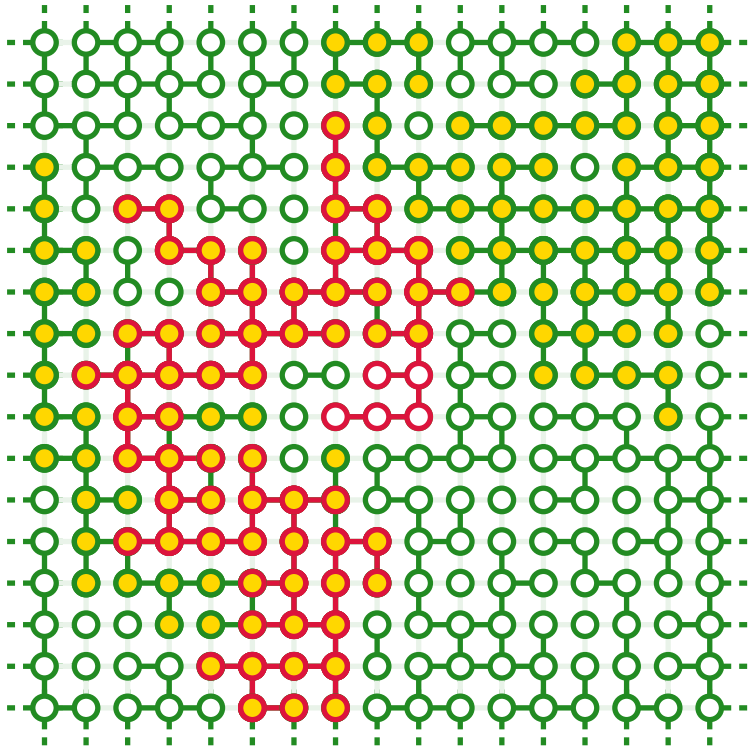

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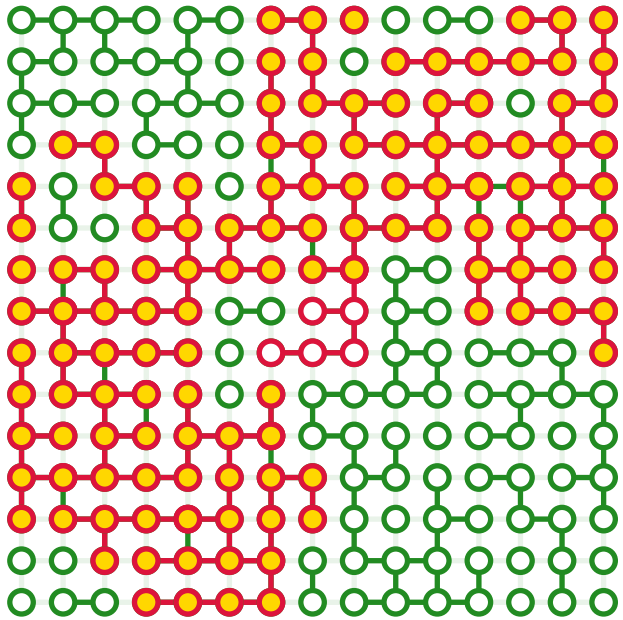

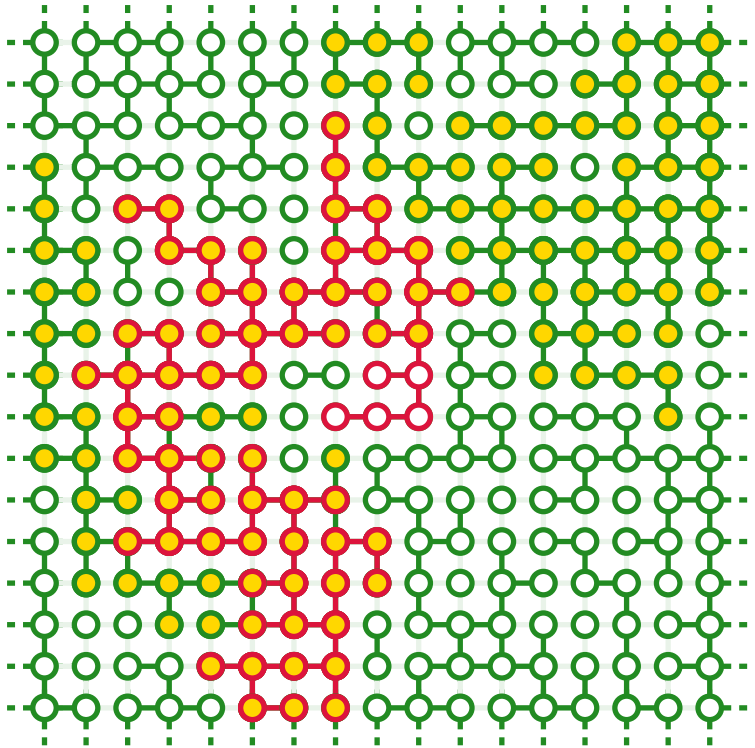

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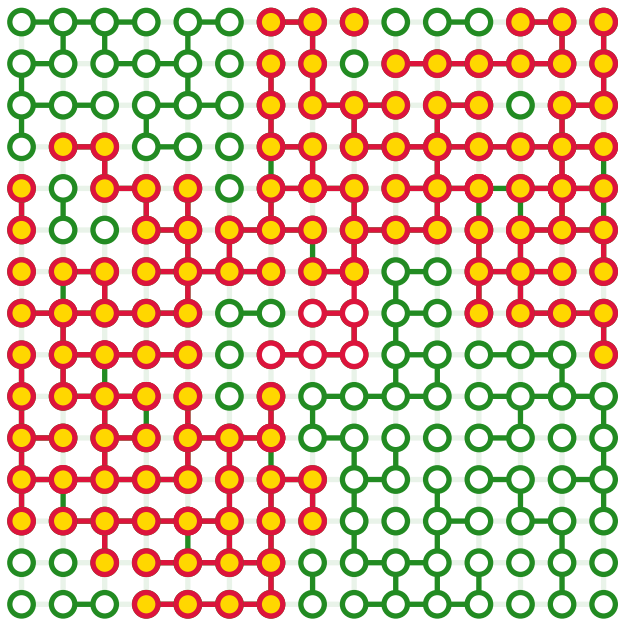
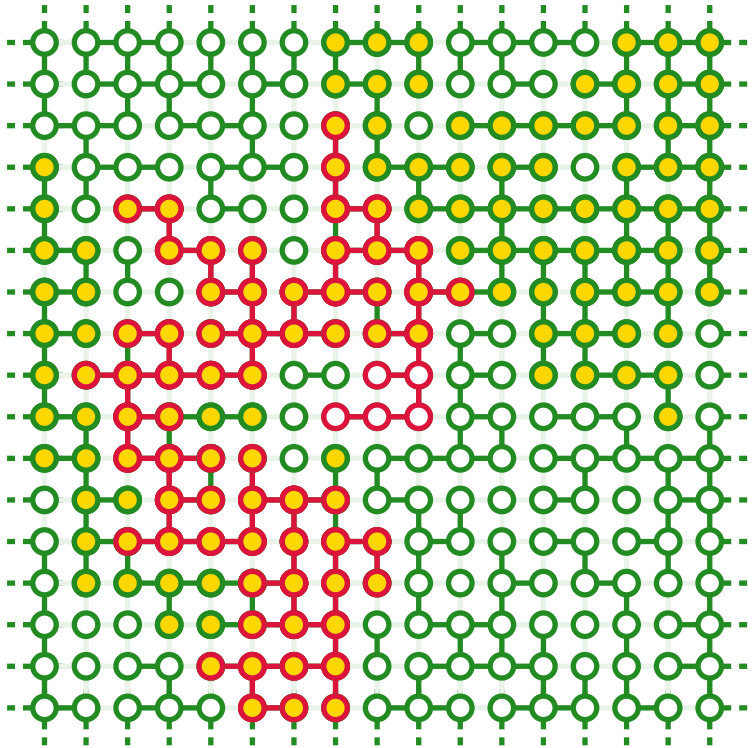
→ CQFD 👍





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